Journal of Natural Sciences and Mathematics, Qassim University, Vol. 3, No. 1, PP 9-20 (June 2009/JumadaII 1430H.)

Weighted Composition Operators on Lorentz - Sequence Spaces

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(Received 23/2/2008; accepted for publication 1/4/2009.)

Abstract. A description of weighted composition operators generated by a sequence and a mapping on Lorentz-sequence spaces $l(p,q),\ \ 1< p\leq\infty,\ \ 1\leq q\leq\infty$ is presented.

Keywords: compact operator, distribution function, Fredholm operator, Lorentz space, Lorentz-sequence space, non-increasing rearrangement , weighted composition operator 2000.
Mathematics Subject Classification: Primary 47B33, 47B38; Secondary 46E30.

1. Introduction

Let $a = < a(n) >_n$ be a complex-valued function defined on the σ-finite measure space (X, A, μ) where $X = \mathbb{N}$, $A = 2^{\mathbb{N}}$, the power set of X and

$\mu =$ counting measure. The distribution function of the complex-valued function $a = < a(n) >_n$ is defined as

$$\mu_a(s) = \mu\{n \in \mathbb{N} : |a(n)| > s\}, s \geq 0.$$

By a^* we mean the *non-increasing rearrangement* of a given as

$$a^*(t) = \inf\{s > 0 : \mu_a(s) \leq t\}, \quad t \geq 0.$$

We can interpret the non-increasing rearrangement of a with $\mu_a(s) < \infty,\ s > 0,$ as a sequence $< a^*_{(n)} >_n$, if we define for $n - 1 \leq t < n,$

$$a^*_{(n)} = a^*(t) = \inf\{s > 0 : \mu_a(s) \leq n - 1\}.$$

Then $a^* = < a^*_{(n)} >_n$ is the sequence $<|a(n)|>_n$ permuted in a decreasing order.

The *Lorentz-sequence space* $l(p,q), 1 < p \leq \infty, 1 \leq q \leq \infty$, is the set of all complex sequences $a = < a(n) >_n$ such that $\| a \|_{(p,q)} < \infty$, where

$$\| a \|_{(p,q)} = \begin{cases} \left\{ \sum_{n=1}^{\infty} (n^{1/p} a^*_{(n)})^q \, n^{-1} \right\}^{1/q}, & 1<p<\infty,\ 1\leq q<\infty \\ \sup_{n\geq 1} n^{1/p} a^*_{(n)}, & 1<p\leq\infty,\ \ q=\infty. \end{cases}$$

The *Lorentz-sequence space* $l(p,q), 1 < p \leq \infty, 1 \leq q \leq \infty$, is a linear space and $\| \cdot \|_{(p,q)}$ is a quasi-norm. Moreover $l(p,q), 1 < p \leq \infty, 1 \leq q \leq \infty$, is complete with respect to the quasi-norm $\| \cdot \|_{(p,q)}$ and $l(p,q), 1 \leq q \leq p < \infty$ with respect to $\| \cdot \|_{(p,q)}$ is a complete normed linear space. Throughout this paper we consider the spaces $l(p,q), 1 < p \leq \infty, 1 \leq q \leq \infty$, with respect to $\| \cdot \|_{(p,q)}$. The l^p - spaces for $1 < p \leq \infty$ are equivalent to the spaces $l(p,p)$. For more details on Lorentz spaces one can refer to [2, 6, 7 and 8] and references therein. Such spaces $l(p,q)$ fall in the category of $L(p,q)$ spaces [7] as well as in the category of functional Banach spaces [6].

A study of the duals, isomorphic l^p $-$subspaces of *Orlicz-Lorentz sequence spaces* $L_{\varphi,w}$ [6 and 8] is made by Kaminska and others. In [10] isomorphic properties of Orlicz-Lorentz sequence spaces are discussed. The *Lorentz-sequence space* $l(p,q)$ coincides with $L_{\varphi,w}$ when $\varphi(t) = t^q$ and the weight sequence $w(n) = n^{(q/p)-1}$. Multiplication and composition operators are studied in various function spaces in [1, 3, 5 and 11]. In [12], these operators are studied on weak Lebesgue space l^p.

Let $T : \mathbb{N} \to \mathbb{N}$ be a mapping and $u = < u(n) >_n$ be a complex sequence (or complex-valued function on $\mathbb{N}$), we define a linear transformation $W_{u,T}$ on the Lorentz-sequence space $l(p,q), 1 < p \leq \infty, 1 \leq q \leq \infty$ into the linear space of all complex sequences by

$$W_{u,T}(f) = u \circ T \cdot f \circ T = < u(T(n)) f(T(n)) >_n$$

where $f = < f(n) >_n \in l(p,q)$. If $W_{u,T}$ is bounded with range in $l(p,q)$, then it is called a weighted composition operator on $l(p,q)$. $B(l(p,q))$ denotes the algebra of all bounded linear operators on $l(p,q)$. An operator $A \in B(l(p,q))$ is said to be Fredholm if it has closed range, $dim(Ker(A))$ and $codim(R(A))$ are finite, where $dim(Ker(A))$ is the dimension of the kernel of A and $codim(R(A))$ is the co-dimension of the range of A, namely the dimension of any subspace complimentary to the range of A.

In this paper we are interested in the study of boundedness and Fredholmness of the weighted composition operator on Lorentz-sequence spaces $l(p,q), 1 < p \le \infty, 1 \le q \le \infty$. Boundedness of the weighted composition operator is characterized. Weighted composition operator with closed range is also characterize

2. Boundedness

The section is devoted to the study of weighted composition operators $W_{u,T}$ $(f \mapsto u \circ T \cdot f \circ T)$ on the space $l(p,q),\ 1 < p \le \infty,\ 1 \le q \le \infty$ induced by a sequence $u = < u(n) >_n$ and a mapping $T : \mathbb{N} \to \mathbb{N}$. Boundedness of $W_{u,T}$ is characterized. Various examples are presented in this section.

Definition 2.1. A sequence $u = < u(n) >_n$ and a mapping $T : \mathbb{N} \to \mathbb{N}$ are said to be finitely related if

$$\mu T^{-1}(\{n\}) < \infty,$$

for all $n \in S = T(\mathbb{N}) \cap \{n : u(n) \neq 0\}$.

Definition 2.2 For some $M > 0$, if the sequence $u = < u(n) >_n$ and the mapping $T : \mathbb{N} \to \mathbb{N}$ satisfy

$$\mu T^{-1}(\{n\}) \le M$$

for all $n \in S = T(\mathbb{N}) \cap \{n : u(n) \neq 0\}$, then u and T are said to be $M-$related.

A necessary condition for the range of $W_{u,T}$ to lie in $l(p,q), 1 < p \le \infty, 1 \le q \le \infty$ is that u and T are finitely related. For if $\mu T^{-1}(\{n\})$ is not finite for some $n \in S$ then $e_n = < e_n(m) >_m$, where

$$e_n(m) = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{otherwise} \end{cases}$$

satisfies $e_n \in l(p,q), 1 < p \le \infty, 1 \le q \le \infty$ with $\| e_n \|_{(p,q)} = 1$ but $W_{u,T} e_n$ does not lie in $l(p,q)$.

Existence of such u and T can be seen by various examples.

Example 2.3. Let $u = < u(n) >_n$ where

$$u(n) = \begin{cases} 1, & \text{if } n = 2; \\ 0, & \text{otherwise} \end{cases}$$

and define $T : \mathbb{N} \to \mathbb{N}$ as

$$T(n) = \begin{cases} n, & \text{if } n \text{ is odd}; \\ 2, & \text{if } n \text{ is even}. \end{cases}$$

Then $S = T(\mathbb{N}) \cap \{n : u(n) \neq 0\} = \{2\}$. Clearly u and T are not finitely related.

Example 2.4. Define $u = < u(n) >_n$ as

$$u(n) = \begin{cases} 0, & \text{if } n \text{ is odd}; \\ n, & \text{if } n \text{ is even} \end{cases}$$

and $T : \mathbb{N} \to \mathbb{N}$ as $T(n) = 2n, \forall\, n \in \mathbb{N}$.

Then $S=\{2n:n\in\mathbb{N}\}$ and $\mu T^{-1}(\{n\})=1$ for all $n\in S$, so that u and T are 1-related.

Example 2.5. Define $u=<u(n)>_n$ as

$$u(n)=\begin{cases}0, & \text{if n is odd;}\\ 1, & \text{if n is even}\end{cases}$$

and $T:\mathbb{N}\to\mathbb{N}$ as

$$T(n)=\begin{cases}1, & \text{if n is odd;}\\ n, & \text{if n is even.}\end{cases}$$

Then u and T are 1-related.

Example 2.6. Define $u=<u(n)>_n$ as $u(n)=1\ \forall\ n\in\mathbb{N}$ and $T:\mathbb{N}\to\mathbb{N}$ as $T(m)=n$ if $2^{n-1}\le m<2^n$. Then $\mu T^{-1}(\{n\})=2^{n-1},\ \forall\ n\in\mathbb{N}\ (=S).$ Hence u and T are finitely related but not M-related for any $M>0$.In case $1\le q<p<\infty$ *or* $q=\infty, 1<p\le\infty,$ for each $n\in\mathbb{N}$, let $m_n=(n^p+1)$ and if $1<p\le q<\infty,$ then $m_n=(n^q+1)$ for each $n\in\mathbb{N}$.

Then in any case, $e_{m_n}=<e_{m_n}(k)>_k$ where

$$e_{m_n}(k)=\begin{cases}1, & \text{if } k=m_n;\\ 0, & \text{otherwise}\end{cases}$$

satisfies $e_{m_n}\in l(p,q)$ with $\|e_{m_n}\|_{(p,q)}=1$.

Now $W_{u,T}e_{m_n}=<u(T(k))e_{m_n}(T(k))>_k$ where

$$u(T(k))e_{m_n}(T(k))=\begin{cases}1, & \text{if } k\in T^{-1}(\{m_n\});\\ 0, & \text{otherwise.}\end{cases}$$

Hence for $1\le q<p<\infty$ *or* $1<p\le q<\infty,$

$$\|W_{u,T}e_{m_n}\|_{(p,q)}^q=1+\frac{1}{2^r}+\cdots+\frac{1}{(\mu T^{-1}(\{m_n\}))^r}\ge(2^{n^q})>n^q\|e_{m_n}\|_{(p,q)}^q,$$

where $r=1-\frac{q}{p}$.

For $q=\infty, 1<p\le\infty$, we have

$$\|W_{u,T}e_{m_n}\|_{(p,q)}=\sup_{k\ge1}k^{1/p}(u\circ T\cdot e_{m_n}\circ T)^*_{(k)}>n\|e_{m_n}\|_{(p,q)}\ .$$

In this example we have seen that finite relatedness of u and T doesn't ensure the boundedness of $W_{u,T}$ on $l(p,q)\ 1<p\le\infty, 1\le q\le\infty$.

Example 2.7. Define $u=<u(n)>_n$ as

$$u(n)=\begin{cases}n, & \text{if n is odd;}\\ 0, & \text{if n is even}\end{cases}$$

and $T:\mathbb{N}\rightarrow\mathbb{N}$ as $T(n)=2n, \forall\, n\in\mathbb{N}$. Then u and T are M-related. Here T is injective, u is unbounded whereas $W_{u,T}(=0)$ is bounded.

Theorem 2.8. Suppose $u=<u(n)>_n$ and $T:\mathbb{N}\rightarrow\mathbb{N}$ are M-related for some $M>0$. Then a necessary and sufficient condition for the boundedness of $W_{u,T}$ is that there exists an $M^*>0$ such that

$$|u(n)|\le M^*$$

for all $n\in S=T(\mathbb{N})\cap\{n:u(n)\ne 0\}$.

Proof. Suppose $u=<u(n)>_n$ and $T:\mathbb{N}\rightarrow\mathbb{N}$ are such that for some $k\in\mathbb{N}$

$$|u(m)|\le k \quad \text{and} \quad \mu T^{-1}(\{m\})\le k, \ \forall \ m\in S.$$

Then for $a=<a(n)>_n\in l(p,q)$ and $t\ge 0$,

$$(u\circ T\cdot a\circ T)^*(kt)=\inf\{s>0:\mu_{u\circ T\cdot a\circ T}(s)\le kt\}$$
$$\le\inf\{s>0:\mu_a(s/k)\le t\}=ka^*(t).$$

Hence

$$\begin{aligned}(u\circ T\cdot a\circ T)^*_{(pk+m)}&=(u\circ T\cdot a\circ T)^*(pk+m-1)\\&=(u\circ T\cdot a\circ T)^*(k(p+\frac{m-1}{k}))\\&\le ka^*(p+\frac{m-1}{k})\\&=ka^*_{(p+1)}, \ \forall \ p\in\mathbb{N}\cup\{0\}, m=1,2,\cdots,k.\end{aligned}$$

This gives for $1<p<\infty,\ 1\le q<\infty$, i.e. for $1\le q<p<\infty \ \ or\, 1<p\le q<\infty$,

$$\begin{aligned}\|W_{u,T}a\|^q_{(p,q)}&=\sum_{n=1}^{\infty}((u\circ T\cdot a\circ T)^*_{(n)})^q \quad n^{(q/p)-1}\\&\le\ k^{2q}\ \|a\|^q_{(p,q)}\end{aligned}$$

and for $q=\infty, 1<p\le\infty$,

$$\begin{aligned}\|W_{u,T}a\|_{(p,q)}&=\sup_{n\ge 1} n^{1/p}(u\circ T\cdot a\circ T)^*_{(n)}\\&\le\ k^2\ \|a\|_{(p,q)}.\end{aligned}$$

Hence $W_{u,T}$ is bounded operator on $l(p,q), 1<p\le\infty, 1\le q\le\infty$.

Conversely, let $W_{u,T}$ be a bounded operator on $l(p,q), 1<p\le\infty, 1\le q\le\infty$ and let $M^*>1$ be such that

$$\|W_{u,T}f\|_{(p,q)}\le M^*\|f\|_{(p,q)} \ \text{ for all } \ f\in l(p,q).$$

In particular, for each $m\in S, e_m\in l(p,q)$ and

$$W_{u,T}e_m =< u(T(k))e_m(T(k)) >_k$$

where

$$u(T(k))e_m(T(k)) = \begin{cases} u(m), & \text{if } k \in T^{-1}(\{m\}); \\ 0, & \text{otherwise} \end{cases}$$

so that

$$\| W_{u,T}e_m \|_{(p,q)} \geq | u(m) | .$$

Thus $| u(m) | \leq \| W_{u,T}e_m \|_{(p,q)} \leq M^*.$ Hence the theorem.

Corollary 2.9. Suppose $u =< u(n) >_n$ and $T : \mathbb{N} \to \mathbb{N}$ are such that $| u(n) | \geq 1 \;\forall\; n \in S.$ Then the linear transformation $W_{u,T}$ on $l(p,q)$ is bounded if and only if

(i): u and T are M-related for some $M > 0$.

(ii): $\exists\, M^* > 0$ such that $| u(n) | \leq M^*$ for all $n \in S$.

Proof. In view of the Theorem 2.8, it is enough to prove that condition (i) holds when $W_{u,T}$ is bounded. If possible (i) doesn't hold. In case $1 \leq q < p < \infty \;\; or \;\; q = \infty, 1 < p \leq \infty,$ then for each $n \in \mathbb{N}$, let $m_n \in S$ be such that

$$\mu T^{-1}(\{m_n\}) > n^p$$

and if $1 < p \leq q < \infty,$ then for each $n \in \mathbb{N}$ take $m_n \in S$ such that

$$\mu T^{-1}(\{m_n\}) > n^q.$$

In any case, $e_{m_n} \in l(p,q).$

Then for $1 < p < \infty, 1 \leq q < \infty$,

$$\begin{aligned} \| W_{u,T}e_{m_n} \|^q_{(p,q)} &= | u(m_n) |^q \, [1 + \frac{1}{2^r} + \cdots + \frac{1}{(\mu T^{-1}(\{m_n\}))^r}] \\ &\geq \begin{cases} \mu T^{-1}(\{m_n\})^{q/p} |u(m_n)|^q, & 1 \leq q < p < \infty \\ \mu T^{-1}(\{m_n\}) |u(m_n)|^q, & 1 < p \leq q < \infty \end{cases} \\ &> n^q \| e_{m_n} \|^q_{(p,q)} \end{aligned}$$

where $\mathrm{r} = 1 - \dfrac{q}{p}.$

Also for $q = \infty, 1 < p \leq \infty,$

$$\begin{aligned} \| \mathrm{W}_{u,T}e_{m_n} \|_{(p,q)} &= \sup_{n \geq 1} n^{1/p} (u \circ T \cdot e_{m_n} \circ T)^*_{(n)} \\ &= (\mu T^{-1}(\{m_n\}))^{1/p} \, | u(m_n) | \\ &> n \| e_{m_n} \|_{(p,q)} \; . \end{aligned}$$

This contradicts the boundedness of $W_{u,T}$. Hence the result.

In the Example 2.7, we have seen that boundedness of u is not necessary condition for the boundedness of $W_{u,T}$ when T is injective. The next corollary states that if T is bijective then it is necessary as well as sufficient condition for the boundedness of $W_{u,T}$.

Corollary 2.10. Suppose $T:\mathbb{N}\to\mathbb{N}$ is a bijective mapping and $u=<u(n)>_n$ is a given sequence. Then $W_{u,T}:l(p,q)\to l(p,q)$ is bounded if and only if u is bounded.

Proof. Injectiveness of T ensures that $u\,\&\,T$ are M -related for $M\geq 1$. Hence we only need to show that if $W_{u,T}$ is bounded then u is bounded.

If $W_{u,T}$ is bounded then, for some $M_0>0$,

$$\|W_{u,T}a\|_{(p,q)}\leq M_0\,\|a\|_{(p,q)},$$

for all $a\in l(p,q)$. For each $n\in\mathbb{N}$, let k_n be the unique natural number such that $T(k_n)=n$, then

$$\|W_{u,T}e_n\|_{(p,q)}\leq M_0,$$

or equivalently $|u(n)|\leq M_0$. Thus u is bounded.

Theorem 2.11. Let $u=<u(n)>_n$ and $T:\mathbb{N}\to\mathbb{N}$ are such that $T(E_\varepsilon)\subseteq E_\varepsilon$ for each $\varepsilon>0,$ where

$$E_\varepsilon=\{n\in\mathbb{N}:|u(n)|>\varepsilon\}.$$

Then u is bounded if $W_{u,T}:l(p,q)\to l(p,q)$ is bounded.

Proof. Suppose $W_{u,T}:l(p,q)\to l(p,q)$ is bounded. In case u is not bounded, then for each $n\in\mathbb{N}$,

$$E_n=\{m\in\mathbb{N}:|u(m)|>n\}$$

is an infinite set. Choose a natural number p^n in $T(E_n)$ and take $F_n=\{p^n\}$. As $W_{u,T}$ is bounded on $l(p,q)$ so u and T are finitely related. Being $p^n\in T(\mathbb{N})\cap\{n:u(n)\neq 0\}$, we find that $T^{-1}(F_n)$ is a non-empty finite set. Now define $a_{F_n}=<a_{F_n}(m)>_m$ where

$$a_{F_n}(m)=\begin{cases}1, & \text{if } m\in F_n;\\ 0, & \text{otherwise.}\end{cases}$$

Then $\|a_{F_n}\|_{(p,q)}=1$. Simple computation shows that

$$\|W_{u,T}(a_{F_n})\|_{(p,q)}\geq|u(p^n)|\quad\geq\quad n=n\,\|a_{F_n}\|_{(p,q)}\ .$$

Thus, for each $n\in\mathbb{N}$ we can find $f_n\in l(p,q)$ satisfying

$$\|W_{u,T}f_n\|_{(p,q)}\geq n\,\|f_n\|_{(p,q)}\ .$$

This contradicts the boundedness of $W_{u,T}$, hence u must be bounded.

Example 2.12. Let $k\in\mathbb{N}$. Define $T:\mathbb{N}\to\mathbb{N}$ as

$$T(n)=\begin{cases}1, & \text{if } n\leq k;\\ n, & \text{if } n>k\end{cases}$$

and $u=<u(n)>_n$ as

$$u(n)=\begin{cases}0, & \text{if } n\neq 1;\\ 1, & \text{if } n=1.\end{cases}$$

Then for each $\varepsilon > 0$,

$$E_{\varepsilon} = \{n \in \mathbb{N} : |u(n)| > \varepsilon\}$$
$$= \begin{cases} \phi, & \text{if } \varepsilon \geq 1; \\ 1, & \text{if } \varepsilon < 1 \end{cases}$$

so that $T(E_{\varepsilon}) \subseteq E_{\varepsilon}$ for each $\varepsilon > 0$. Here u & T are k-related, u is bounded so that $W_{u,T}$ is a bounded operator although T is not injective.

3. Closed Range

In this section we characterize the weighted composition operators on $l(p,q)$ having closed range and Fredholm weighted composition operators on $l(p,q), 1 < p \leq \infty, 1 \leq q \leq \infty$.

Theorem 3.1. Let $u = < u(n) >_n$ and $T : \mathbb{N} \to \mathbb{N}$ are M-related for some $M > 0$. Then $W_{u,T} \in B(l(p,q)),\ 1 < p \leq \infty,\ 1 \leq q \leq \infty$ has closed range if and only if there exists a $\delta > 0$ such that

$$|u(n)| > \delta$$

for all $n \in S = T(\mathbb{N}) \cap \{n : u(n) \neq 0\}$.

Proof. Let $u = < u(n) >_n$ and $T : \mathbb{N} \to \mathbb{N}$ are M-related for some $M > 0$ with $W_{u,T} \in B(l(p,q))$. Let there exists a $\delta > 0$ such that $|u(n)| > \delta$ for all $n \in S$. Let $f^{(k)} = < f^{(k)}(n) >_n \in l(p,q)$ be such that $W_{u,T} f^{(k)} \to f$ as $k \to \infty$, where $f = < f(n) >_n \in l(p,q)$. Then

$$\| W_{u,T} f^{(n)} - W_{u,T} f^{(m)} \|_{(p,q)} \to 0 \text{ as } n, m \to \infty.$$

For each $k \in \mathbb{N}$, put $g^{(k)} = < g^{(k)}(n) >_n$ where

$$g^{(k)}(n) = \begin{cases} f^{(k)}(n), & \text{if } n \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Then $|g^{(k)}(n)| \leq |f^{(k)}(n)|,\ \forall\ n$ and we find that $g^{(k)} \in l(p,q)$ for each $k \in \mathbb{N}$. Also $W_{u,T} g^{(k)} = W_{u,T} f^{(k)}$ as for each $n \in \mathbb{N}$,

$$(u \circ T \cdot g^{(k)} \circ T)(n) = \begin{cases} u(T(n)) f^{(k)}(T(n)), & \text{if } T(n) \in S; \\ 0, & \text{otherwise.} \end{cases}$$
$$= (u \circ T \cdot f^{(k)} \circ T)(n).$$

Moreover $|(g^{(n)} - g^{(m)})(k)| \leq |(f^{(n)} - f^{(m)})(k)|$ for all $k \in \mathbb{N}$. Hence

$$\| g^{(n)} - g^{(m)} \|_{(p,q)} \leq \| f^{(n)} - f^{(m)} \|_{(p,q)}.$$

Now we claim that for each $k \in \mathbb{N}$,

$$\delta (g^{(n)} - g^{(m)})^*_{(k)} \leq (W_{u,T}(g^{(n)} - g^{(m)}))^*_{(k)}.$$

For each $k \in S$ say $k = T(s_k)$ for some $s_k \in \mathbb{N}$, we have

$$\delta\,|(f^{(n)}-f^{(m)})(k)| \;=\; \delta\,|(f^{(n)}-f^{(m)})(T(s_k))|$$
$$\le\; |(W_{u,T}(f^{(n)}-f^{(m)}))(s_k)|.$$

This gives for all $k \in \mathbb{N}$,

$$\delta(g^{(n)}-g^{(m)})^*_{(k)} \;\le\; (W_{u,T}(f^{(n)}-f^{(m)}))^*_{(k)}$$
$$= (W_{u,T}(g^{(n)}-g^{(m)}))^*_{(k)}.$$

Therefore for $1<p<\infty, 1<q<\infty$, i.e. for $1\le q<p<\infty \quad or\; 1<p\le q<\infty$,

$$\|g^{(n)}-g^{(m)}\|^q_{(p,q)} \;=\; \sum_{k=1}^{\infty}((g^{(n)}-g^{(m)})^*_{(k)})^q k^{(q/p)-1}$$
$$\le \sum_{k=1}^{\infty}\frac{1}{\delta^q}((W_{u,T}(g^{(n)}-g^{(m)})^*_{(k)}))^q k^{(q/p)-1}$$
$$= \frac{1}{\delta^q}\|W_{u,T}g^{(n)}-W_{u,T}g^{(m)}\|^q_{(p,q)}$$
$$= \frac{1}{\delta^q}\|W_{u,T}f^{(n)}-W_{u,T}f^{(m)}\|^q_{(p,q)}$$
$$\to 0 \quad \text{as} \quad n,m\to\infty$$

and also for $q=\infty, 1<p\le\infty$,

$$\|g^{(n)}-g^{(m)}\|_{(p,q)} = \sup_{k\ge 1} k^{1/p}((g^{(n)}-g^{(m)})^*_{(k)})$$
$$\le \frac{1}{\delta}\|W_{u,T}g^{(n)}-W_{u,T}g^{(m)}\|_{(p,q)}$$
$$= \frac{1}{\delta}\|W_{u,T}f^{(n)}-W_{u,T}f^{(m)}\|_{(p,q)}$$
$$\to 0 \quad \text{as} \quad n,m\to\infty.$$

As $l(p,q),\; 1<p\le\infty,\; 1\le q\le\infty$ is complete and $<g^{(n)}>_n$ is a Cauchy sequence in $l(p,q)$ so we find $g\in l(p,q)$ such that $g^{(n)}\to g$ as $n\to\infty$, and hence

$$W_{u,T}f^{(n)} \;=\; W_{u,T}g^{(n)} \to\; W_{u,T}g \;\text{ as }\; n\to\infty.$$

Thus $f=W_{u,T}g$ so that $W_{u,T}$ has closed range.

Conversely, if $W_{u,T}$ has closed range then for some $\varepsilon>0$

$$\|W_{u,T}f\|_{(p,q)} \ge \varepsilon\|f\|_{(p,q)}$$

for all $f\in l_{pq}(S)$, where

$$l_{pq}(S)=\{a=<a(n)>_n\in l(p,q): a(n)=0\;\forall\, n\in\mathbb{N}\;,\;\; S\}.$$

In case $S=\phi$, then nothing to prove. Suppose $S\ne\phi$. Consider the case $1\le q<p<\infty$. We claim that

$$|u(m)| \geq \frac{\varepsilon}{M^{1/q}}$$

for all $m \in S$. If possible

$$|u(m)| < \frac{\varepsilon}{M^{1/q}}$$

for some $m \in S$, then $e_m \in l_{pq}(S)$ and

$$\| W_{u,T} e_m \|^q_{(p,q)} \quad \leq \quad |u(m)|^q \, M < \varepsilon^q,$$

which is a contradiction. Hence

$$|u(m)| \quad \geq \quad \frac{\varepsilon}{M^{1/q}} \quad \text{for all} \quad m \in S.$$

For $1 < p \leq q < \infty, \ or \quad q = \infty, 1 < p \leq \infty,$ we find that

$$|u(m)| \quad \geq \quad \frac{\varepsilon}{M^{1/p}} \quad \text{for all} \quad m \in S.$$

Hence in any case we can find a $\delta > 0$ such that $|u(n)| > \delta$ for all $n \in S = T(\mathbb{N}) \cap \{n : u(n) \neq 0\}$. $\square$

Theorem 3.2. Suppose $u = < u(n) >_n$ and $T : \mathbb{N} \to \mathbb{N}$ are M-related for some $M > 0$. Then $W_{u,T} \in B(l(p,q)), 1 < p \leq \infty, 1 \leq q \leq \infty$ is Fredholm if and only if

(i): $\mathbb{N} \setminus S$ is a finite set.

(ii): $E = \{n \in \mathbb{N} : \mu T^{-1}(\{n\}) \geq 2\}$ is a finite set.

(iii): there exists $\varepsilon > 0$ such that $|u(n)| \quad > \varepsilon \ \forall \ n \in S$.

Proof. Suppose $W_{u,T} \in B(l(p,q))$ is Fredholm. As $Ker W_{u,T} = l_{pq}(\mathbb{N} \setminus S)$, where

$$l_{pq}(\mathbb{N} \setminus S) = \{a = < a(n) >_n \in l(p,q) : a(n) = 0 \ \forall \ n \in S\},$$

so $\mathbb{N} \setminus S$ is a finite set.

Also, if the set $E = \{n \in \mathbb{N} : \mu T^{-1}(\{n\}) \geq 2\}$ is an infinite set then for each $k \in E$, let $n_k, m_k \in \mathbb{N}$ are such that $T(n_k) = T(m_k), n_k \neq m_k$.

For each $k \in E$, define $f_k = < f_k(m) >_m$ where

$$f_k(m) = \begin{cases} 1, & \text{if } m = n_k; \\ -1, & \text{if } m = m_k; \\ 0, & \text{if } m \neq n_k, m_k. \end{cases}$$

Then $f_k \in l(p,q) \setminus R(W_{u,T})$. Moreover $\{f_k : k \in E\}$ is linearly independent hence $l(p,q) \setminus R(W_{u,T})$ is infinite dimensional, which is a contradiction. Therefore E is a finite set. Condition (iii) is obvious in view of the Theorem 3.1. Converse is easy to prove.

Acknowledgement

The authors are thankful to the referees for their valuable suggestions and critical comments which led to a lot of improvement over the original version of the paper.

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مؤثرات التكوين المرجح على فضاءات متتابعة لورينتز

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قسم الرياضيات ، دلهي ، الهند

(قدم للنشر في ٢٠٠٨/٢/٢٣م، وقبل للنشر في ٢٠٠٩/٤/١م)

ملخص البحث. يقدم البحث وصفاً لمؤثرات التكوين المرجح والمولدة بواسطة متتابعة والمنقولة على فضاءات متتابعة لورينتز $l(p,q)\ ,\ 1 \le p \le \infty$.