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Modified Product Block By Block For The Second Kind Volterra Integral Equations With Singular Kernel

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Abstract: The block-by -block method is a good self-starting method for solving Volterra Integral equations of the second kind with continuous kernel. In this paper two modifications are suggested to develop block by block method to solve Volterra integral equations with weakly singular kernels. The idea of the product technique with higher order is used for block-by block method, which is the first modification, while the second modification is using the method on the graded nodes. Implementation and testing of the considered algorithms have been done. The results showed that this method gives good results compared with others methods on equal spaced nodes.

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1. Introduction

Volterra integral equations arise most natural in certain types of time-dependent problems whose behavior at time t depends not only on the state at that time, but also on the states at pervious times as in renewal equation. Volterra integral equations have applications in history –dependent problems, in the system theory, in heat conduction and diffusion, see [7] and [12]. Mathematical modeling processes in biological and physical applications lead quite frequently to Volterra integral equation of the second kind of which has following form:

$$y(x) = f(x) + \int_0^x p(x,t)\,k\big(x,t,y(t)\big)\,dt, \quad x \in [0,T] \tag{1}$$

where p(x, t) is the singular term,
k (x, t, y (t)) is smooth on 0 ≤ t ≤ x ≤ T and
y(x) satisfies y(0) = f (0). In this paper, the following singularity is considered:

$$p(x,t) = (x-t)^{-\alpha}\, t^{\alpha-1}, \quad 0 < \alpha < 1, \tag{2}$$

where the function *p(x, t)* has two types of singularities, the first is fixed at $t = 0$ and the other is movable singularity along the line $x = t$. The commonly encountered values of α are (0 and ½), but other (rational) values (e.g., $\alpha = 2/3$) are also known to occur, for example in the modeling of the flow of a hot gas through a metallic tube, with reaction arising at the walls of the tube; (see Atkinson [1] and Claus [9]).

The solution of the Volterra Integral equation as Fredholm integral Equation is called Block-by-Block method, and the product technique is a good method to solve singular integral Equations. In this paper the idea of product technique is used to modify block-by- block method and using it to solve Volterra Integral equations with weakly singular kernels.

Many papers using Fredholm integral equation technique to solve Volterra integral equations. Orsi [13] introduced a numerical approach for solving Volterra integral equations of the second kind when the kernel contains a mild singularity. The idea of that approach was to use the technique of Fredholm equations to find the starting values of equation (1). A convergence result in [13] was illustrated by numerical examples.

The second kind Volterra integral equations with weakly singular kernels typically have solutions which are not smooth near the initial point of the integration interval. Brunner [7] - [8] used an adaptation of the analysis originally developed for nonlinear weakly singular Fredholm Integral equations, and presented a complete discussion of the optimal (global and local) order of convergence of piecewise polynomial collocation methods on graded grids for nonlinear Volterra integral equations with algebraic or logarithmic singularities in their kernels.

Kolk and Padas used a pricewise polynomial in collocation methods to solve Volterra integral equations with weakly singular kernels [11]. They estimated the global convergence and gave numerical results.

Attia [4] used block method with Bool's quadrature to solve Volterra integral equation of the second kind on the graded nodes. Here the high order product technique is used to modify the block by block method. This modification is used to solve Volterra integral equation of the second kind with weakly singular kernels.

2. Graded Mesh

Volterra integral equation of the second kind with singular kernel of the form (1) is assumed to satisfy the conditions for a unique solution, Linz [12]. The interval [0, T] is divided into N = 4M subintervals. The nodes as reported by Attia [2-4], are chosen to satisfy the following:

$$0 = t_0 < t_1 < t_2 \,.......< t_{N-1} < t_N = T \tag{3}$$

Besides, the even nodes are found from:

$$t_{4k} = \left[\frac{4k}{N}\right]^{\beta} T = \left[\frac{k}{M}\right]^{\beta} T, \; k = 0,1,2,\ldots,M \tag{4}$$

and the width of each subinterval is given by:

$$h_{4k} = \frac{t_{4k+4} - t_{4k}}{4}, \quad k = 0,1,...,M-1. \tag{5}$$

The others nodes are obtained from:

$$t_{4k+j} = t_{4k} + jh_{4k}, \quad j = 1,2,3 \;\; and \quad k = 0,1,...,M-1, \tag{6}$$

$$x_k = t_k, \;\; k = 0,1,...,4M. \tag{7}$$

3. Weights and Error for Product Integration

The product-integration method is based on approximating the smooth function k*(x, t, y)* by a polynomial, more precisely, the integral is approximated by:

$$\int_0^{x_{4i}} p(x_{4i},t)\; k(x_{4i},t,y(t))\,dt \cong \sum_{j=0}^{4} \bar{w}_j\, k(x_{4i},t_j,y_j) \tag{8}$$

where $x_{4i} = t_{4i},\ 1, 2, ..., N$ and $p(x_{4i},t) = (x_{4i} - t)^{-\alpha}\, t^{\alpha-1}$ which is the singular part. The weights (w_j) depend on the quadrature point's x_i.

Theorem 3.1

The singular integral of the following form:

$$\int_0^{x_{4l}} \left(x_{4l} - t\right)^{-\alpha} t^{\alpha-1} f\left(t\right) dt$$

is approximated by:

$$\sum_{j=0}^{4} \bar{w}_j f\left(t_j\right)$$

Where the values of weights are:

$$\bar{w}_0 = \frac{\pi}{9\sin\alpha\pi}\left[4\alpha^4 - 16\alpha^3 + 29\alpha^2 - 26\alpha + 9\right],$$

$$\bar{w}_1 = \frac{\pi}{9\sin\alpha\pi}\left[-16\alpha^4 + 48\alpha^3 - 56\alpha^2 + 24\alpha\right],$$

$$\bar{w}_2 = \frac{\pi}{9\sin\alpha\pi}\left[24\alpha^4 - 48\alpha^3 + 30\alpha^2 - 6\alpha\right], \qquad (9)$$

$$\bar{w}_3 = \frac{\pi}{9\sin\alpha\pi}\left[-16\alpha^4 + 16\alpha^3 - 8\alpha^2 + 8\alpha\right],$$

$$\bar{w}_4 = \frac{\pi}{9\sin\alpha\pi}\left[4\alpha^4 + 5\alpha^2\right].$$

Proof

To compute the weights (w_j), the following system of equations is solved:

$$\sum_{j=0}^{4} \bar{w}_j f\left(t_j\right) = \int_0^{x_{4l}} \left(x_{4l} - t\right)^{-\alpha} t^{\alpha-1} f\left(t\right) dt, \quad f\left(t\right) = t^k, \quad k = 0,1,\ldots,4 \qquad (10)$$

The error bound is computed from (10) when f (t) = t^5 and the weights in (9) are used, this gives:

$$\bar{E}_p \le \frac{16\pi h^5 \alpha}{15\sin\alpha\pi}\left[8\alpha^4 - 20\alpha^3 + 30\alpha^2 - 25\alpha + 7\right]. \qquad (11)$$

when α = ½ the value of error of the product integration E_p will be equal zero, this means that E_p of order *$O(h^6)$*, but the value of the error term at α = 2/3 will be equal:

$$\bar{E}_p \le -\frac{704\pi h^5}{729\sqrt{3}} = O\left(h^5\right). \qquad (12)$$

4. Bool's Quadrature

The Bool's quadrature formula states that [10]:

$$\int_0^{x_4} \varphi(x)\,dx = \frac{2h}{45}\left[7\varphi_0 + 32\varphi_1 + 12\varphi_2 + 32\varphi_3 + 7\varphi_4\right] \qquad (13)$$

Furthermore, there exists a value c with c $\in$ (0, x_4), such that the error term E_B takes the form:

$$E_B = -\frac{8h^7}{945}\varphi^{(6)}(c). \tag{14}$$

In Simpson's block–by-block method, two equations in two unknowns are solved in each block or in each stage with an error of order $O(h^4)$, [6]. In our case where Bool's quadrature is used for block-by-block method, actually we solve four equations in four unknown but the error is less than the error of Simpson's block method it will be of order $O(h^6)$ and the method is considered as the extrapolation of Simpson's method. The system of equations of Bool's block-by-block method is:

$$y_{4k+1} = f_{x_{4k+1}} + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=0}^{4} w_l k\left(x_{4k+1}, t_{4j-1}, y\left(t_{4j-1}\right)\right) + \sum_{l=0}^{4} \overline{w}_l k\left(x_{4k+1}, t_{4k+l/4}, y\left(t_{4k+l/4}\right)\right) \tag{15}$$

$$y_{4k+2} = f_{x_{4k+2}} + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=0}^{4} w_l k\left(x_{4k+2}, t_{4j-1}, y\left(t_{4j-1}\right)\right) + \sum_{l=0}^{4} \overline{w}_l k\left(x_{4k+2}, t_{4k+l/2}, y\left(t_{4k+l/2}\right)\right), \tag{16}$$

$$y_{4k+3} = f_{x_{4k+3}} + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=0}^{4} w_l k\left(x_{4k+3}, t_{4j-1}, y\left(t_{4j-1}\right)\right) + \sum_{l=0}^{4} \overline{w}_l k\left(x_{4k+3}, t_{4k+3l/4}, y\left(t_{4k+3l/4}\right)\right), \tag{17}$$

$$y_{4k+4} = f_{x_{4k+4}} + \sum_{j=1}^{k} 2h_{4j-4} \sum_{l=4}^{0} w_l k\left(x_{4k+4}, t_{4j-1}, y\left(t_{4j-1}\right)\right) + \sum_{l=0}^{4} \overline{w}_l k\left(x_{4k+4}, t_{4k+l}, y\left(t_{4k+l}\right)\right), \tag{18}$$

Where k = 0, 1, 2,..., M-1, and

$$\left[w_l\right] = \left[\frac{7}{45} \quad \frac{32}{45} \quad \frac{12}{45} \quad \frac{32}{45} \quad \frac{7}{45}\right], \quad l = 0,1,\cdots,4 \tag{19}$$

At each stage, equations (15) – (18) have to be solved simultaneously for the unknown y_{4k+1}, y_{4k+2}, y_{4k+3}, y_{4k+4}, so that we obtain a block of unknowns at each subinterval $[x_{4k}, x_{4(k+1)}]$, $k = 0, 1, \ldots, M-1$. The values of $y_{4k+1/4}$, $y_{4k+1/2}$, $y_{4k+3/4}$, $y_{4k+3/2}$, $y_{4k+9/4}$ are computed by interpolation of a polynomial of degree four , using y_{4k+1}, y_{4k+1}, y_{4k+2}, y_{4k+3}, *and* y_{4k+4}, as follows:

$$y_{4k+1/4} = \frac{1}{2048}\left[1155y_{4k} + 1540y_{4k+1} - 990y_{4k+2} + 420y_{4k+3} - 77y_{4k+4}\right], \tag{20}$$

$$y_{4k+1/2} = \frac{1}{128}\left[35y_{4k} + 140y_{4k+1} - 70y_{4k+2} + 28y_{4k+3} - 5y_{4k+4}\right], \tag{21}$$

$$y_{4k+3/4} = \frac{1}{2048}\left[195y_{4k} + 2340y_{4k+1} - 702y_{4k+2} + 260y_{4k+3} - 45y_{4k+4}\right], \tag{22}$$

$$y_{4k+3/2} = \frac{1}{128}\left[-5y_{4k} + 60y_{4k+1} + 90y_{4k+2} - 20y_{4k+3} + 3y_{4k+4}\right], \tag{23}$$

$$y_{4k+9/4} = \frac{1}{2048}\left[35y_{4k} - 252y_{4k+1}1890y_{4k+2} + 420y_{4k+3} - 45y_{4k+4}\right], \tag{24}$$

5. Test Example

The following example is used to test the technique and is taken from [2].

$$y(x) = x^2\left[2-\lambda\Gamma(2+\alpha)\Gamma(1-\alpha)\right] + \lambda\int_0^x (x-t)^{-\alpha} t^{\alpha-1} y(t)dt, \tag{25}$$

Equation (25) has an exact solution y (x) = $2x^2$. Where $\Gamma(\alpha)$ is the Gamma function which satisfies $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$ and $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin(\alpha\pi)$.

Results of the root mean square errors, maximum errors and its positions for this example at different cases are given in the following table.

Where E_{rms}: Root Mean Square Error,

E_M: Maximum Error,

X_{EM}: Position of maximum error and

E_R: is the ratio between E_{rms} of Bool's and Simpson's products

Table (1) Results of, the root mean square errors, maximum errors and its position

λ	N	Simpson's product block-by-block				Bool's product block-by-block			E_R
		β	E_{rms}	X_{EM}	E_M	E_M	X_{EM}	E_{rms}	
-5	4	0.95	2.38×10^{-2}	1.000	3.74×10^{-2}	4.14×10^{-4}	1.000	2.42×10^{-4}	0.010
	8	0.90	1.73×10^{-3}	1.000	3.24×10^{-3}	4.43×10^{-5}	1.000	2.06×10^{-5}	0.012
	16	0.80	1.25×10^{-4}	0.259	2.11×10^{-4}	4.01×10^{-6}	1.000	1.78×10^{-6}	0.014
	32	0.80	9.76×10^{-6}	1.000	1.57×10^{-5}	3.36×10^{-7}	1.000	1.46×10^{-7}	0.149
0.2	4	1.10	2.58×10^{-3}	0.733	5.72×10^{-3}	5.94×10^{-5}	0.750	2.78×10^{-5}	0.011
	8	0.95	1.66×10^{-4}	0.880	3.28×10^{-4}	3.33×10^{-6}	0.875	1.54×10^{-6}	0.009
	16	0.95	9.69×10^{-6}	0.940	2.05×10^{-5}	2.23×10^{-7}	0.937	9.70×10^{-8}	0.010
	32	0.90	6.63×10^{-7}	0.972	1.30×10^{-6}	1.59×10^{-8}	0.969	6.87×10^{-9}	0.010
1	4	1.05	3.94×10^{-2}	1.000	5.12×10^{-2}	5.44×10^{-4}	1.000	4.12×10^{-4}	0.010
	8	1.10	3.01×10^{-3}	0.880	6.09×10^{-3}	6.53×10^{-5}	0.875	3.57×10^{-5}	0.012
	16	1.15	2.13×10^{-4}	0.929	5.18×10^{-4}	6.07×10^{-6}	0.937	3.04×10^{-6}	0.014
	32	1.15	1.54×10^{-5}	0.964	4.02×10^{-5}	5.27×10^{-7}	0.969	2.54×10^{-7}	0.016

6. Conclusions

The product methods give good results when using it on graded mesh compared with equal spaced nodes at the same number of subintervals. We conclude to use product block-by-block on graded mesh. The equal spaced nodes can be considered as a special case from the graded mesh.

In this paper the weights for product technique are computed, and the technique is used to modify the block-by-block method. The modified technique is used to solve Volterra integral equations with weakly singular kernels. The kernel with two types of singularities, the first at t = 0 and the second moves along the line x = t, the results of experiment showed that:

1. The product block-by-block method is a good method to solve Volterra integral equations with singular kernels.

2. The value of β that gives min error in the case of graded mesh: $\beta \in (0.7, 1.5)$

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تطوير طريقة مضروبات الحزم لحل معادلة فولترا التكاملية من النوع الثاني ذات النواه المفردة

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ملخص البحث. تعتبر طريقة الحل حزمة بحزمة من الطرق الجيدة ذاتية البدء لحل معادلة فولترا التكاملية من النوع الثاني ذات النواة المتصلة. وفي هذا البحث تم اقتراح تطوير للطريقة لحل معادلة فولترا من النوع الثاني ذات النواه المفردة باقتراحين:

أولا: تم استنتاج أوزان طريقة المضروبات عالية الرتبة ومن ثم استخدامها مع طريقة الحل حزمة بحزمة.

ثانيا: تهيئة الطريقة للاستخدام علي شبكة متدرجة.

وقد تم اختبار الطريقة وتم مقارنة النتائج بطريقة حزم سيمسون والمقارنة بالشبكة ذات المسافات المتساوية.