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On a Compendious Structure on a Differentiable Manifold

Geeta Verma

Department of Applied Sciences ,Saraswati Institute of Technology & Management ,
Unnao. (U.P.), INDIA
E-mail : geeta_verma1@yahoo.com

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Abstract. In recent years several structures, notable almost contact atructure [2], [6], [8], almost r-contact structure [4],[15], almost paracontact structure [9], almost r-paracontact structure [2], almost contact hyperbolicstructure [14] and almost r-contact hyperbolic structure [5] have been defined and studied on a differentiable manifold by many geometers. Some generalized structures, including almost $(\varepsilon_1, \varepsilon_2)$-contact structure [10], almost $(\varepsilon_1, \varepsilon_2)$-r-contact structure [11], [12] and unified structure [1], [13] have also been defined. In this paper, We define and study a compendious structure having the following structures as its special cases

Keywords: compendious structure/integrability and Nijenhuis tensor.

1. Introduction

We first define a compendious structure Σ on a differentiable manifold as follows:

Definition 1 : Let M be an m-dimentional differentiable manifold admitting a tensor field F of type (1,1),linearly independent vector fields (T_x) and 1-form (A_x), x = 1,2,…., r , r < m, such that

$$F(T_x) = 0 \tag{1}$$

$$F^2 X = ea^r X + cA^r(X)T_x \tag{2}$$

where e,c take values ± 1 and a^r is a (complex) constant. We define the structure $\Sigma \equiv (F, T_x, A^x)$ to be a compendious structure on M and the pair (M, Σ) or simply M to be a compendious structure manifold.

Agreement 1: In the above and in what follows the indices x,y,z,.... run over (1,2,…..r)

And the equation s containing X,Y,Z,……hold for arbitrary vector fields unless otherwise stated.

Theorem 1 : If M be a compendious structure manifold, then

$$A_x F = 0 \tag{3}$$

$$A^x(T_y) = -eca^r \delta_y^x \tag{4}$$

$$\text{rank}(F)=m-r \tag{5}$$

δ_y^x being Kronecker's symbol.

Now introduce a metric M.

Definition 2 : On a compendious structure manifold (M, Σ) let a metric g be introduced such that

$$g(FX, FY) = a^r g(X,Y) + ec\sum_x A^x(X)A^x(Y) \tag{6}$$

we define $(\Sigma, g) \equiv (F, T_x, A^x, g)$ to be a compendious metric structure and M equipped

with such a metric structure to be a compendious metric structure manifold. The above metric g is said to a metric associated to the compendious structure on M.

Setting X=T_x an immediate consequence is that A^x is the covariant form of T_x, that is

$$A^x(Y) = g(T_x, Y) \tag{7}$$

Theorem 2 : On a compendious structure manifold (M,Σ) there always exists a metric g ,given by (6).

Proof : Let h' be any Riemannian metric on M and let h be defined by

$$a^r h(X,Y) \overset{def}{=} -ec\left[h'(F^2X, F^2Y) + \sum_x A^x(X)A^x(Y)\right]$$

Then $h(T_x, Y) = A^x(Y)$ and it is easy to checkthat h is a metric. Now let us defined g by

$$2a^r g(X,Y) \overset{def}{=} h(FX, FY) + a^r h(X,Y) - ec\sum_x A^x(X)A^x(Y)$$

Again g is clearly a metric and the relation

$$\begin{aligned} 2a^r g(FX,FY) &= a^r h(FX,FY) + h(ea^r X + cA^x(X)T_x, ea^r Y + cA^x(Y)T_x) \\ &= a^r h(FX,FY) + a^{2r} h(X,Y) + eca^r \sum_x A^x(X)A^x(Y) \\ &= 2a^{2r} g(X,Y) + 2eca^r \sum_x A^x(X)A^x(Y) \end{aligned}$$

implying (6). However, the metric g is , of course, not unique.

Theorem 3 : On a comoendious metric structure manifold (M, Σ, g) the following relations hold good :

$$g(T_x, FX) = 0 \tag{8}$$

$$g(FX, Y) = eg(X, FY) \tag{9}$$

The proof is obvious.

Using (1) and (2), it is easy to verify the following result.

Theorem 4 : Let (F,T_x,A^x) and $(F,T_x,\overline{A}^x)\left[resp.(F,\overline{T}_x,A^x)\right]$ be two compendious structure on a differentiable manifold M , then we have $A^x=\overline{A}^x\left[resp. T_x=\overline{T}_x\right]$.

Thus we see that two compendious structure having same F and same $(T_x)[resp.(A^x)]$ on a differentiable manifold M always includes another compendious structure on M. So we can prove the following theorem.

Theorem 5 : A compendious structure on a differentiable manifold M is not unique.

Proof : Let H be arbitrary non singular tensor field of type (1,1) on M. Defining

$$\overline{F}\overset{def}{=}H^{-1}FH,\qquad \overline{A}^x\overset{def}{=}A^xH,\qquad \overline{T}_x\overset{def}{=}H^{-1}(T_x) \tag{10}$$

it can easily seen that $(\overline{F},\overline{T}_x,\overline{A}^x)$ is also a compendious structure on M. Moreover if g is an associated metric to the structure (F,T_x,A^x) on M, then a metric $\overline{g}$ on M defined by

$$\overline{g}(X,Y)\overset{def}{=}g(HX,HY) \tag{11}$$

provides an associated metric to the structure $(\overline{F},\overline{T}_x,\overline{A}^x)$ on M.

We can state this fact as foolows.

Corollary (1) : A compendious metric structure on a differentiable manifold is not unique.

2. Existence of a compendious structure

Let λ be an eigen value of F corresponding to an eigen vector P.We now consider the following two possible cases.

Case I P is linearly independent of (T_x).Then (2) implies that $(\lambda^2-ea^r)P=cA^x(P)T_x$. Hence $\lambda=\pm\sqrt{ea^r}$ and $A^x(P)=0$.

Case II P is a linear combination of (T_x). Then F(P) = 0 that is $\lambda=0$.Therefore, there are r eigen values 0.

Since M is of dimension m and rank(F) = m-r, there are, say, r eigen values 0, s eigenvalues $+\sqrt{ea^r}$ and m-r-s eigenvalues $-\sqrt{ea^r}$. Let L, K and N denote the distributions corresponding to the eigenvalues 0, $+\sqrt{ea^r}$ and $-\sqrt{ea^r}$ respectively.

Lemma(1)

The distributions L,K, and N are complementary distributions generated by the complementary projection operators l, k and n defined by

$$l \overset{def}{=} (a^r I - eF^2)a^{-r} \tag{12}$$

$$2k \overset{def}{=} (eF^2 + dF)a^{-r} \tag{13}$$

and

$$2n \overset{def}{=} (eF^2 - dF)a^{-r} \tag{14}$$

respectively, where I is the identity tensor field and $d = e\sqrt{ea^r}$.

Proof: We see that l+k+n = I.We also have

$$l^2 = (a^{2r} I + F^4 - 2ea^r F^2)a^{-2r} = (a^{2r} I + ea^r F^2 - 2ea^r F^2)a^{-2r} = l$$

$$k^2 = (F^4 + d^2 F^2 + 2edF^3)\frac{1}{4}a^{-2r} = (ea^r F^2 + ea^r F^2 + 2da^r F)\frac{1}{4}a^{-2r} = k$$

And similarly $n^2 = n$. Again, we get

$$2lk = (ea^r F^2 + da^r F - F^4 - edF^3)a^{-2r} = (ea^r F^2 + da^r F - ea^r F^2 - da^r F)a^{-2r} = 0$$

$$2l\,n = (ea^r F^2 - da^r F - F^4 + edF^3)a^{-2r} = (ea^r F^2 - da^r F - ea^r F^2 + da^r F)a^{-2r} = 0$$

$$4kn = F^4 - d^2 F^2 = ea^r F^2 - ea^r F^2 = 0$$

Consequently *l*, k, n are complementary projection operators. Moreover

$$Fl = (a^r F - eF^3)a^{-r} = (a^r F - a^r F)a^{-r} = 0$$

$$Fk = (eF^3 + dF^2)\frac{1}{2}a^{-r} = (a^r F + e\sqrt{ea^r} F^2)\frac{1}{2}a^{-r} = \sqrt{ea^r}(eF^2 + dF)\frac{1}{2}a^{-r} = k\sqrt{ea^r}$$

and similarly $Fn = -n\sqrt{ea^r}$. We also get $k + n = a^{-r} eF^2$.

Now it remains to show that L, K, and N are the complementary distributions generated by the complementary projection operators l, k, and n that is $L = \{lX; X \in \chi(M)\}$, $K = \{kX; X \in \chi(M)\}$ and $N = \{nX; X \in \chi(M)\}$.

Let $Z \in L$ .Then, since 0 is the eigen value for L, we have FZ = 0.Also, since Z = *l*Z+kZ+nZ,we get 0 = FZ = *Fl*Z+DkZ+FnZ = 0 + $\sqrt{ea^r}\,kZ - \sqrt{ea^r}\,nZ$ or kZ-nZ = 0. But k+n = $a^{-r}eF^2$; therefore kZ+ nZ = 0. Hence kZ = 0 and nZ = 0 and thus Z=*l*Z, that is $L \subset \{lX; X \in \chi(M)\}$.

Conversely, let Z = *l* X, then FZ = F*l*X = 0 which shows that $Z \in L$, that is $\{lX; X \in \chi(M)\} \subset L$. Thus $L = \{lX; X \in \chi(M)\}$.

Again, if $Z \in K$,then since $\sqrt{ea^r}$ is the eigen value for K.We have $\sqrt{ea^r}Z = FZ = FlZ + FkZ + FnZ = 0 + \sqrt{ea^r}kZ - \sqrt{ea^r}nZ$ or Z = Kz – n Z. Also $kZ + nZ = ea^{-r}F^2Z = Z$. Thus Z = k Z, that is $K \subset \{kX; X \in \chi(M)\}$.On the otherhand, let Z = k X, then $FZ = FkX = \sqrt{ea^r}kX = \sqrt{ea^r}Z$. Thus $Z \in K$; that is , $\{kX; X \in \chi(M)\} \subset K$. Hence $K = \{kX; X \in \chi(M)\}$. Similarly, we can prove that $N = \{nX; X \in \chi(M)\}$.

Agreement(2): In what follows the indices $i, j[resp. i', j']$ run over $\{1,......, s\}[resp. \{1,......., m-r-s\}]$.Now we are in a position to prove the main theorem of this section.

Theorem(6) : A necessary and sufficient condition for M to admit a compendious structure is that there exists complementary projection operators l, k and n which bring together the complementary distributions L, K and N of dimensions r,s and m-r-s respectively, which together span the manifold.

Proof: The necessary part follows from **Lemma(1)**.For sufficient part,let $(T_x, U_i, U_{i'})$ be a set such that (T_x), (U_i) and $(U_{i'})$ are the basis vectors in L,K and N respectively, and let $(-eca^{-r}A^x, V^i, V^{i'})$ be the inverse set.

Therefore, we get

$$\begin{aligned} &-eca^{-r}A^x(T_y) = \delta^x_y,\ A^x(U_i) = 0,\ A^x(U_{i'}) = 0, \\ &V^i(T_x) = 0, V^i(U_j) = \delta^i_j, V^i(U_{j'}) = 0 \\ &V^{i'}(T_x) = 0, V^{i'}(U_j) = 0,\ V^{i'}(U_{j'}) = \delta^{i'}_{j'} \end{aligned} \tag{15}$$

and

$$V^i(X)U_i + V^{i'}(X)U_{i'} - eca^{-r}A^x(X)T_x = X$$

or

$$ea^rV^i(X)U_i + ea^rV^{i'}(X)U_{i'} - cA^x(X)T_x = ea^rX \tag{16}$$

$$FX = \sqrt{ea^r}V^i(X)U_i + \sqrt{ea^r}V^{i'}(X)U_{i'} \tag{17}$$

we have $F(T_x) = 0$ and

$$F^2 X = \sqrt{ea^r} V^i (FX) U_i + \sqrt{ea^r} V^{i'} (X) U_i$$
$$= \sqrt{ea^r} V^j (\sqrt{ea^r} V^i (X) U_i + \sqrt{ea^r} V^{i'} (X) U_{i'}) U_j + \sqrt{ea^r} V^{j'} (\sqrt{ea^r} V^i (X) U_i$$
$$+ \sqrt{ea^r} V^{i'} (X) U_{i'}) U_{j'}$$
$$= ea^r V^i (X) U_i + ea^r V^{i'} (X) U_{i'}$$
$$= ea^r X + cA^x (X) T_x$$

Thus (F, T_x, A^x) defines a compendious structure on M.

3. Integrability Conditions

Let us recall some relations of the previous section as follows :

$$lk = kl = l\,n = nl = kn = nk = o, \tag{18}$$

$$l^2 = l,\ k^2 = k,\ n^2 = n, \tag{19}$$

$$Fl = lF = 0 \tag{20}$$

$$Fk = kF = k\sqrt{ea^r} \tag{21}$$

$$Fn = nF = -n\sqrt{ea^r} \tag{22}$$

$$F^2 l = 0,\ \ F^2 k = ea^r k,\ \ F^2 n = ea^r n \tag{23}$$

Lemma (2) : If [F, F] is the Nijenhuis tensor of F, then

$$l[F, F](lX, lY) = 0 \tag{24}$$

$$k[F, F](kX, kY) = 0 \tag{25}$$

$$n[F, F](nX, nY) = 0 \tag{26}$$

$$l[F, F](kX, kY) = ea^r l[kX, kY] \tag{27}$$

$$l[F,F](nX,nY) = ea^r l[nX,nY] \quad (28)$$

$$k[F,F](lX,lY) = ea^r k[lX,lY] \quad (29)$$

$$k[F,F](nX,nY) = 4ea^r k[nX,nY] \quad (30)$$

$$n[F,F](lX,lY) = ea^r n[lX,lY] \quad (31)$$

$$n[F,F](kX,kY) = 4ea^r n[kX,kY] \quad (32)$$

Proof : The Nijenhuis tensor [F,F] of F is defined by

$$[F,F](X,Y) = [FX,FY] - F[FX,FY] - F[X,FY] + F^2[X,Y] \quad (33)$$

But $k + n = e^{-1}F^2 a^{-r}$, therefore we get

$$[F,F](X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + ea^r k[X,Y] + ea^r n[X,Y] \quad (34)$$

On putting in (34) nX and nY in place of X and Y respectively, operating the whole equation by k and using (18), (23), we get (30).
Similarly we get (19)-((29), (31), (32).
Finally , we prove main theorem of this section.
Theorem (7) : The compendious structure manifold M is completely integrable if and only if

$$\begin{aligned}[F,F](X,Y) = {} & [F,F](lX,lY) + [F,F](lX,nY) + [F,F](kX,lY) \\ & + [F,F](kX,lY) + [F,F](kX,nY) + [F,F](nX,lY) \\ & + [F,F](nX,kY)\end{aligned} \quad (35)$$

Proof: It is well known that any distribution D is integrable if and only if $[X,Y] \in D$ for all $X,Y \in D$. Thus,the distribution L is integrable if and only if

$$k[lX,lY] = 0 \quad (36)$$

$$n[lX,lY] = 0 \quad (37)$$

Equivalently, from (29) and (31), we have

$$k[F,F](lX,lY) = 0 \quad (38)$$

$$n[F,F](lX,lY)=0 \tag{39}$$

The distribution K is integrable if and only if

$$l[kX,kY]=0 \tag{40}$$

$$n[kX,kY]=0 \tag{41}$$

and , equivalently,

$$l[F,F](kX,kY)=0 \tag{42}$$

$$n[F,F](kX,kY)=0 \tag{43}$$

Similarly, the distribution N is integrable if and only if

$$l[nX,nY]=0 \tag{44}$$

$$k[nX,nY]=0 \tag{45}$$

and equivalently

$$l[F,F](nX,nY)=0 \tag{46}$$

$$k[F,F](nX,nY)=0 \tag{47}$$

The Nijenhuis tensor [F,F] of F can be written in the form

$$[F,F](X,Y)=(l+k+n)[F,F]((l+k+n)X,(l+k+n)Y)$$

Expanding right hand and using (24), (26), (38), (39), (40), (41), we get (35).

4. Special Cases

The structure of this paper generalizes many known structures which may be obtained by taking particular values of a^r, e, c, r . We list these particular cases by giving different values to a^r, e, c, r , writing structural equations corresponding to (2), (4), (6), (9) and discussing the details.

Case 1 : $a^r = 1, e \equiv \varepsilon_1 = \pm 1, c \equiv \varepsilon_2 = \pm 1$). Almost $(\varepsilon_1, \varepsilon_2) - r -$ contact Riemannian structure K.D.Singh and R.K. Agnihotri and R. Singh.

$$F^2 X = \varepsilon_1 X + \varepsilon_2 A^x(X) T_x$$
$$A^x(T_y) = -\varepsilon_1 \varepsilon_2 \delta_y^x$$
$$g(FX, FY) = g(X, Y) + \varepsilon_1 \varepsilon_2 \sum_x A^x(X) A^x(Y)$$
$$g(FX, Y) = \varepsilon_1 g(X, FY)$$

Case 2 : $(a^r = 1, e \equiv \varepsilon_1 = \pm 1, c \equiv \varepsilon_2 = \pm 1, r = 1)$. Almost $(\varepsilon_1, \varepsilon_2) -$ contact Riemannian structure I. Sato.

$$F^2 X = \varepsilon_1 X + \varepsilon_2 A(X) T,$$
$$A(T) = -\varepsilon_1 \varepsilon_2$$
$$g(FX, FY) = g(X, Y) + \varepsilon_1 \varepsilon_2 A(X) A(Y)$$
$$g(FX, Y) = \varepsilon_1 g(X, FY)$$

The existence theorem already has been discussed for cases 1, 2. Now integrability conditions can be deduced from this paper.

Agreement (3) : In the above and in what follows, when $r = 1$, (A^i, T_i) will be identified by (A, T).

Case 3 : $(a^r = 1, e = -1, c = 1)$. Almost r-contact Riemannian structure [4], [7], [15].

$$F^2 X = -X + A^x(X) T_x,$$
$$A^x(T_y) = \delta_y^x$$
$$g(FX, FY) = g(X, Y) - \sum A^x(X) A^x(Y)$$
$$g(FX, Y) = -g(X, FY)$$

Case 4 : $(a^r = 1, e = -1, c = 1, r = 1)$. Almost contact Riemannian structure [2], [6], [8].

$$F^2X = -X + A(X)T,$$
$$A(T) = 1$$
$$g(FX, FY) = g(X, Y) - A(X)A(Y)$$
$$g(FX, Y) = -g(X, FY)$$

In cases 3, 4 the dimension of K becomes equal to the dimension of N and hence , in case of almost contact manifold, the manifold becomes odd dimensional.

Case 5 : $(a^r = 1, e = 1, c = -1)$. Almost r-paracontact Riemannian structure [3].

$$F^2X = X - A^x(X)T_x,$$
$$A^x(T_y) = \delta^x_y$$
$$g(FX, FY) = g(X, Y) - \sum_x A^x(X)A^x(Y)$$
$$g(FX, Y) = g(X, FY)$$

Case 6 : $(a^r = 1, e = 1, c = -1, r = 1)$. Almost paracontact Riemannian structure [9].

$$F^2X = X - A(X)T,$$
$$A(T) = 1$$
$$g(FX, FY) = g(X, Y) - A(X)A(Y)$$
$$g(FX, Y) = g(X, FY)$$

All the results can be deduced for cases 5,6 by putting appropriate vales for a^r, e, c, r .

Cases 7 : $(a^r = -1, e = -1, c = 1)$. Almost r-contact hyperbolic Riemannian structure [5].

$$F^2 X = X + A^x(X)T_x,$$
$$A^x(T_y) = -\delta_y^x$$
$$g(FX, FY) = -g(X,Y) - \sum_x A^x(X)A^x(Y)$$
$$g(FX, Y) = -g(X, FY)$$

Case 8 : $(a^r = -1, e = -1, c = 1, r = 1)$. Almost contact hyperbolic Riemannian structure [14].

$$F^2 X = X + A(X)T,$$
$$A(T) = -1$$
$$g(FX, FY) = -g(X,Y) - A(X)A(Y)$$
$$g(FX, Y) = -g(X, FY)$$

To the best of my knowledge , existence and integrability in cases 7, 8 have not studied so far.

Case 9: $(a^r$ replaced by $-a^r, e = -1, c = 1, r = 1)$. Unified metric structure [1] , [13].

$$F^2 X = a^r X + A(X)T,$$
$$A(T) = -a^r$$
$$g(FX, FY) = -a^r g(X,Y) - A(X)A(Y)$$
$$g(FX, Y) = -g(X, FY)$$

Putting $(\varepsilon_1, \varepsilon_2) = (-1,1)$, $(\varepsilon_1, \varepsilon_2) = (1,-1)$ and $(\varepsilon_1, \varepsilon_2) = (1,1)$ in case 2 we get almost contact Riemannian structure , almost paracontact Riemannian structure and almost contact hyperbolic structure (but not almost contact hyperbolic Riemannian structure) respectively. In fact , when $(\varepsilon_1, \varepsilon_2) = (1,1)$. we have

$$g(FX, FY) = g(X,Y) + A(X)A(Y), \quad g(FX, Y) = g(X, FY)$$

which does not coincide with the metric of case 8. However, if we take a particular case of the compendious metric structure by setting $a^r = -1,\ e = -1,\ c = 1,\ r = 1$, it would be possible to find an almost contact hyperbolic Riemannian structure [14].

The unified metric structure [1], [13] only unifies an almost contact Riemannian structure [2],[6],[8] and an almost contact hyperbolic Riemannian structure [14]. However, if we take a particular case of compendious metric structure by setting $e = -1,\ c = 1$ and a^r replaced by $-a^r$, it would be possible to find a metric structure which unifies an almost contact Riemannian structure [2],[6],[8] an almost r-contact Riemannian structure [4], [7], [15], an almost contact hyperbolic Riemannian structure [14] and an almost r-contact hyperbolic structure [5].

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