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Higher Order Product Block by Block Method for Nonlinear Volterra Integral Equations of the Second Kind with Weakly Singular Kernel

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Abstract: In this paper product block technique is modified to solve nonlinear Volterra integral equations of the second kind when the kernel contains mild singularity. The product block by block method is modified by using two suggestions. Firstly we use Bool's quadrature rule which has high order of convergence. Secondly the Bool's block is adapted to use on graded nodes. To illustrate the effectiveness of the developed method a number of numerical examples are presented and a comparison with Simpson's product block by block is made.

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1. Introduction

Volterra equations arise most naturally in certain types of time-dependent problems whose behavior at time t depends not only on the state at that time, but also on the states at pervious times as renewal equation. Volterra integral equations also have many applications in history-dependent problems, in system theory and in heat conduction and diffusion [1].

We consider nonlinear Volterra integral equations of the second kind of the form:

$$\varphi(t) = g(t) + \int_0^t p(t,s) K(t,s,\varphi(s)) ds, \quad t \in R^+ \tag{1}$$

where the kernel p(t, s) is weakly singular and the given functions g and K are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $\varphi \in C\,[0,T]$ (see, for instance, [2] and [3])

Orsi [4] introduced a numerical approach for solving Volterra integral equations of the second kind when the kernel contains a mild singularity. The idea of this approach uses the technique of Fredholm equations to find starting values of
G. M.

equation (1). The author gave a convergence result and also presented numerical examples to illustrate the performance and efficiency of the method.

Second-kind Volterra integral equations with weakly singular kernels typically have solutions which are non-smooth near the initial point of the interval of integration. Brunner [5] used an adaptation of the analysis originally developed for nonlinear weakly singular Fredholm integral equations, and presented a complete discussion of the optimal (global and local) order of convergence of piecewise polynomial collocation methods on graded grids for nonlinear Volterra integral equations with algebraic or logarithmic singularity in their kernels.

Different algorithms on graded nodes are used to solve different types of Volterra integral equations of the second kind [6], [7] and [8].

Blyth et el [9] used Walsh function method to find a numerical solution of the second kind Volterra integral equations as Fredholm type. Volterra integral equations are rewritten as Fredholm integral equations with appropriately modified kernels. But in this technique, it is noted that the Walsh function solution method has low order of convergence, if we compared it with the quadrature methods.

Karapetyants et el [10] considered Volterra integral equation of the form:

$$\varphi^m(t) = g(t) + \alpha(t) \int_0^t k(t-s) b(s) \varphi(s) ds, \quad 0 < t < d \le \infty \tag{2}$$

where m > 1 and real-valued functions α(t), k(u), b(s) and g(t). If b(s) = 1 and m > 0 this equation is arising in applications, e.g. in water percolation [11] and [12] and in the nonlinear theory of wave propagation [13]. When m > 1, and g(t) = 0, equation (2) may have a nontrivial solution φ (t) [14]. Equation (2) with α(t) = b(s) =1, 0 < m < 1 and a continuous kernel k(u) was considered in [15], where some results were given on the uniqueness of its solution ϕ(t) in some spaces of continuous or integrable functions. Such a problem for equation (2) with m < 0 and non-increasing kernel k(u) in the class of almost decreasing functions was studied in [16] with α(t) = b(s) = 1. Lower estimates and asymptotic properties near zero for the solution φ (t) of equation (2) with m > 1 were obtained in [17] provided that α(t), k(u) and g(t) have power asymptotic behavior near zero.

In the following section we state the basic theories of the existence and uniqueness of the solution of the nonlinear Volterra integral equation of the second kind on the interval [0, T] for some T > 0. Section (3) gives a short survey on literature on Simpson's product block by block and the investigated modified product block by block for Volterra equations with mild singularity of the second kind. In section (4) we shall give the results of an illustrated examples and the conclusion.

2. Preliminaries

This section is devoted to introduce the main theories of existence and uniqueness of the solution of Volterra integral equation of the second type with weakly singular kernel, (see [18]-[20]).

2.1. Existence and Uniqueness

Consider the linear Volterra integral equation of the second in the form:

$$\varphi(t) = g(t) + \left(Z_\alpha \varphi(t)\right), \quad t \in [0, T], \quad (0 < \alpha < 1). \tag{3}$$

With initial condition: φ(0) = g(0).

Here the Volterra integral operator Z_α (from C [0, T] into C [0, T] is characterized by a weakly singular kernel; i. e., we have

$$(Z_\alpha f)(t) = \int_0^t (t-s)^{-\alpha} K(t,s) f(s) ds \tag{4}$$

where K(t, s) is a smooth on the domain S = {(t, s): 0 ≤ s ≤ t ≤ T}.

2.2 Lemma

Let K(t, s) ∈ $C^m(S)$, then for 0 < α < 1, the resolvent kernel R = R(t, s; α) associated with the kernel F(t, s; α) = $(t\text{-}s)^{-\alpha}$ K (t, s) in (3) has the form:

$$R(t,s;\alpha) = (t-s)^{-\alpha} Q(t,s;\alpha), \quad (t,s) \in S \tag{5}$$

with Q∈ C(S) is given by:

$$Q(t,s;\alpha)=\sum_{n=1}^{\infty}\Phi_n(t,s;\alpha)(t-s)^{(n-1)(1-\alpha)} \tag{6}$$

This series converges uniformly and absolutely on S, and we have $\Phi_n \in C^m(S)$ for all $n \geq 1$.

This result is easily established by computing the iterated kernels of F(t, s; α), the resulting recurrence relation for $\{\Phi_n\}$ (with Φ_1 = K(t, s)) yields the smoothness result for these functions.

2.3 Theorem

In equation (3) assume that the functions $g(t) \in C^m[0, T]$ and $K(t,s) \in C^m(S)$, with $m \geq 0$, then the unique φ solution to (3) has the form:

$$\varphi(t)=g(t)+\sum_{n=1}^{\infty}\Psi_n(t,\alpha)(t)^{n(1-\alpha)} \tag{7}$$

The functions $\{\Psi_n(t, \alpha)\}$ are given by:

$$\Psi_n(t,\alpha)=\int_0^1(1-\zeta)^{n(1-\alpha)-1}\Phi_n(t,\zeta t;\alpha)\,g(\zeta t)\,d\zeta \tag{8}$$

and they satisfy $\Psi_n \in C^m[0, T]$), ($n \geq 1$).

It follows from the Volterra theory that the solution of (3) can be expressed in the form:

$$\varphi(t)=g(t)+\int_0^t R(t,s;\alpha)g(s)\,ds \tag{9}$$

2.4 Corollary

Consider that K(t,s) and g(t) in (3) are constants on their respective domains:

K(t,s) = k, g(t) = G. Then the corresponding solution is:

$$\varphi(t)=G\left\{1+\sum_{n=1}^{\infty}\frac{\left[k.\Gamma(1-\alpha)t^{(1-\alpha)}\right]^n}{\Gamma(1+n(1-\alpha))}\right\}$$

$$=G.E_{1-\alpha}\left(k\Gamma(1-\alpha)t^{(1-\alpha)}\right), \tag{10}$$

where $E_{1-\alpha}$ (denotes the Mittag-Leffler function (see [21]).

3. Method of Solution

One popular method for solving Volterra integral equation of the second kind is the quadrature methods. The error of the quadrature methods depends on the rule of integration and the position of the mesh points. The following technique is general for determining the nodes, for example if $\beta = 1$, we have the case of equal space nodes (uniform nodes). If $\beta \neq 1$, we have the case of graded nodes. For every rule, the nodes are adapted as follows

3.1 Graded Nodes for Product Simpson's Block-by-Block

The interval [0, T] is divided into N = 2M subintervals. The nodes are chosen to satisfy:

$$0 = t_0 < t_1 < t_2 \ldots\ldots < t_{N-1} < t_N = T \tag{11}$$

Besides, the even nodes are found from:

$$t_{2k} = \left[\frac{2k}{N}\right]^{\beta} T = \left[\frac{k}{M}\right]^{\beta} T,\ k = 0,1,2,\ldots,M. \tag{12}$$

The odd nodes are computed as the following:

$$t_{2k+1} = \frac{t_{2k+2} + t_{2k}}{2}, \quad k = 0,1,\ldots,M-1. \tag{13}$$

The width of each subinterval is given by:

$$h_k = t_{k+1} - t_k, \quad k = 0,1,\ldots,2M-1. \tag{14}$$

$$s_k = t_k,\ k = 0,1,\ldots,2M. \tag{15}$$

3.2 Product Simpson's Block by Block Methods

The product block by block method is essentially an extrapolation procedure and has the advantage of being self starting. This method can be adapted to use on the graded nodes (see [6]). The product trapezoidal method has low order of convergence method, and the other methods as product Simpson's requires one or more starting values. We shall follow Linz description of the method (see [1]). The so-called product block-by-block methods are generalization of the well-known implicit Runge-Kutta methods for ordinary differential equations and one finds the latter term also used in connection with integral equations. The idea behind the product block-by-block methods is quite general, but is most easily understood by considering a specific case.

Let us assume that we have decided to use product Simpson's rule as the numerical integration formula. If we know φ_1, then, we could simply compute φ_2 by:

$$\varphi_2 = g(t_2) + \omega_{2,0}K(t_2,t_0,\varphi_0) + \omega_{2,1}K(t_2,t_1,\varphi_1) + \omega_{2,2}K(t_2,t_2,\varphi_2) \quad (16)$$

To obtain a value of φ_1, we introduce another point $t_{½} = h_0/2$ and the corresponding value $\varphi_{½}$, then use product Simpson's rule with φ_0, $\varphi_{½}$ and φ_1 to give:

$$\varphi_1 = g(t_1) + \omega_{1,0}K(t_1,t_0,\varphi_0) + \omega_{1,1/2}K(t_1,t_{1/2},\varphi_{1/2}) + \omega_{1,1}K(t_1,t_1,\varphi_1) \quad (17)$$

With

$$\omega_{n,j} = \int_0^{t_n} p(t_n,s)\, l_{n,j}\, ds \quad (18)$$

Where L_{ij} are the fundamental polynomial defined as:

$$L_{ij} = \frac{(t-t_{i,o})\cdots(t-t_{i,j-1})(t-t_{i,j+1})\cdots(t_1-t_{i,m})}{(t_{i,j}-t_{i,o})\cdots(t_{i,j}-t_{i,j-1})(t_{i,j}-t_{i,j+1})\cdots(t_{i,j}-t_{i,m})} \quad (19)$$

The unknown value of $\varphi_{½}$ can approximated by the quadratic interpolation, using the values of φ_0, φ_1 and φ_2, that is, $\varphi_{½}$ is replaced by

$$\varphi_{1/2} \approx \frac{1}{8}\left[3\varphi_0 + 6\varphi_1 - \varphi_2\right] \quad (20)$$

Equations (16) and (17) are a pair of simultaneous equations for φ_1 and φ_2. For sufficiently small h a unique solution exists and can be obtained by Newton's method. The general process should now be clear; for m = 0, 1,…, (N/2). The approximate solution can be computed, on a graded mesh described before for product block-by-block method, by:

$$\varphi_{2m+1} = g_{t_{2m+1}} + (1-\delta_{0m})\sum_{j=0}^{m-1}\sum_{k=0}^{2} w_{2m+1,2j+k}K(t_{2m+1},t_{2j+k},\varphi_{2j+k}) + \frac{h_{2m}^{(1-\alpha)}}{2(2-\alpha)(3-\alpha)}.$$
$$.\left[(1-\alpha)K(t_{2m+1},t_{2m},\varphi_{2m}) + 4K(t_{2m+1},t_{2m+1/2},\varphi_{2m+1/2}) + \frac{(1+\alpha)}{(1-\alpha)}K(t_{2m+1},t_{2m+1},\varphi_{2m+1})\right] \quad (21)$$

and

$$\varphi_{2m+2} = g_{t_{2m+2}} + (1-\delta_{0m})\sum_{j=0}^{m}\sum_{k=0}^{2} w_{2m+2,j+k}K(t_{2m+2},t_{2j+k},\varphi_{2j+k}), \quad (22)$$

$$\varphi_{2m+1/2} \approx \frac{1}{8}\left[3\varphi_{2m} + 6\varphi_{2m+1} - \varphi_{2m+2}\right], \quad m = 0,1,2,\ldots,N/2 \quad (23)$$

$$\omega_{n,2j} = \frac{T_2^{(2-\alpha)}\left[(1+\alpha)h_{2j} - 2T_1\right] + T_1^{(1-\alpha)}\left[2T_1^2 + 3(\alpha-3)T_1h_{2j} + 2h_{2j}\left(6-5\alpha+\alpha^2\right)\right]}{2h_{2j}^2(1-\alpha)(2-\alpha)(3-\alpha)} \quad (24)$$

$$\omega_{n,2j+1}=\frac{2T_1^{(2-\alpha)}\left[T_1+(\alpha-3)h_{2j}\right]+2T_2^{(2-\alpha)}\left[h_{2j}(\alpha-1)-T_1\right]}{h_{2j}^2(\alpha-1)(\alpha-2)(\alpha-3)} \tag{25}$$

$$\omega_{n,2j+2}=\frac{T_2^{(1-\alpha)}\left[2T_1^2-T_1(3\alpha-1)h_{2j}+2h_{2j}^2(\alpha-1)^2\right]-T_1^{(2-\alpha)}\left[2T_1+(\alpha-3)h_{2j}\right]}{2h_{2j}^2(\alpha-1)(\alpha-2)(\alpha-3)} \tag{26}$$

Where

$$T_1=\left(t_n-t_{2j}\right),\ and\quad T_2=\left(t_n-t_{2j+2}\right). \tag{27}$$

3.3 Graded Nodes for Product Bool's Block-by-Block

In this subsection two suggestions are made. The first is using the Bool's quadrature rule which is considered an extrapolation technique of Simpson's rule and the second is using the graded nodes. The Bool's quadrature formula takes the form:

$$\int_0^{t_4}\varphi(t)\,dt=\frac{h}{45}\left[7\varphi_0+32\varphi_1+12\varphi_2+32\varphi_3+7\varphi_4\right]+O\left(h^6\right). \tag{28}$$

The interval [0, T] is divided into N = 4M subintervals. The nodes are chosen to satisfy:

$$0=t_0<t_1<t_2\,.......<t_{N-1}<t_N=T \tag{29}$$

The even nodes are found from:

$$t_{4k}=\left[\frac{4k}{N}\right]^{\beta}T=\left[\frac{k}{M}\right]^{\beta}T,\ k=0,1,2,\ldots,M. \tag{30}$$

The width of each subinterval is:

$$h_{4k}=\frac{t_{4k+4}-t_{4k}}{4},\quad k=0,1,...,M-1. \tag{31}$$

The other nodes are computed as the following:

$$t_{4k+j}=t_{4k}+jh_{4k},\quad j=1,2,3\ \ and\ k=0,1,...,M-1, \tag{32}$$

$$s_k=t_k,\ \ k=0,1,...,4M. \tag{33}$$

3.4 Modified Product Block by Block Method

According to the previous methods and the work in [6-8] the accuracy and rate of convergence can be increased by dealing with Bool's ideas but take care with the singularity at s = t and using the interpolation technique. In the product Simpson's block-by-block method two equations in two unknowns are solved in each stage. In the case of product Bool's block-by-block method really we solve four equations in four unknowns in each stage but the method is considered an extrapolation technique of product Simpson's block-by-block. The product Bool's block-by-block method takes the form; for m = 0, 1, …, (M-1);

$$\varphi_{4m+1} = g\left(t_{4m+1}\right) + \left(1-\delta_{om}\right)\sum_{j=1}^{m-1}\sum_{k=0}^{4}\omega_{4m+1,4j+k}K\left(t_{4m+1}, t_{4j+k}, \varphi_{4j+k}\right) + \sum_{k=0}^{4}\varpi_{4m+1,4m+k}K\left(t_{4m+1}, t_{4m+k/4}, \varphi_{4m+k/4}\right), \tag{34}$$

$$\varphi_{4m+2} = g\left(t_{4m+2}\right) + \left(1-\delta_{om}\right)\sum_{j=1}^{m-1}\sum_{k=0}^{4}\omega_{4m+2,4j+k}K\left(t_{4m+2}, t_{4j+k}, \varphi_{4j+k}\right) + \sum_{k=0}^{4}\varpi_{4m+2,4m+k/2}K\left(t_{4m+2}, t_{4m+k/2}, \varphi_{4m+k/2}\right), \tag{35}$$

$$\varphi_{4m+3} = g\left(t_{4m+3}\right) + \left(1-\delta_{om}\right)\sum_{j=1}^{m-1}\sum_{k=0}^{4}\omega_{4m+3,4j+k}K\left(t_{4m+3}, t_{4j+k}, \varphi_{4j+k}\right) + \sum_{k=0}^{4}\varpi_{4m+3,4m+3k/4}K\left(t_{4m+2}, t_{4m+3k/4}, \varphi_{4m+3k/4}\right), \tag{36}$$

$$\varphi_{4m+4} = g\left(t_{4m+4}\right) + \left(1-\delta_{om}\right)\sum_{j=1}^{m-1}\sum_{k=0}^{4}\omega_{4m+4,4j+k}K\left(t_{4m+4}, t_{4j+k}, \varphi_{4j+k}\right) + \sum_{k=0}^{4}\varpi_{4m+3,4m+k}K\left(t_{4m+4}, t_{4m+k}, \varphi_{4m+k}\right), \tag{37}$$

Where

$$\varpi_{4m+k,4m+\frac{j}{4}} = \frac{32\left(2^{4j}\right)}{3h_{4m}^{4}}\int_{0}^{kh_{4m}} s\left(\frac{jh_{4m}}{4}-s\right)\left(\frac{2jh_{4m}}{4}-s\right)\left(\frac{3jh_{4m}}{4}-s\right)\left(jh_{4m}-s\right)^{1-\alpha}ds, \tag{38}$$

$$\left(s-\frac{jh_{4m}}{4}\right)\text{ is skipped},\ k=1,2,...,4, \text{and } j=0,1,....4$$

At each stage, equations (34) – (37) have to be solved simultaneously for the unknown φ_{4k+1}, φ_{4k+2}, φ_{4k+3}, φ_{4k+4}, so that we obtain a block of unknowns at each subinterval $[x_{4k}, x_{4(k+1)}]$, k = 0, 1, ..., M-1. The values of $\varphi_{4k+1/4}$, $\varphi_{4k+1/2}$, $\varphi_{4k+3/4}$, $\varphi_{4k+3/2}$, $\varphi_{4k+9/4}$ are computed by interpolation of a polynomial of degree four, using φ_{4k+1}, φ_{4k+1}, φ_{4k+2}, φ_{4k+3}, and φ_{4k+4}, as follows:

$$\varphi_{4k+1/4} = \frac{1}{2048}\left[1155\varphi_{4k} + 1540\varphi_{4k+1} - 990\varphi_{4k+2} + 420\varphi_{4k+3} - 77\varphi_{4k+4}\right], \qquad (39)$$

$$\varphi_{4k+1/2} = \frac{1}{128}\left[35\varphi_{4k} + 140\varphi_{4k+1} - 70\varphi_{4k+2} + 28\varphi_{4k+3} - 5\varphi_{4k+4}\right], \qquad (40)$$

$$\varphi_{4k+3/4} = \frac{1}{2048}\left[195\varphi_{4k} + 2340\varphi_{4k+1} - 702\varphi_{4k+2} + 260\varphi_{4k+3} - 45\varphi_{4k+4}\right], \qquad (41)$$

$$\varphi_{4k+3/2} = \frac{1}{128}\left[-5\varphi_{4k} + 60\varphi_{4k+1} + 90\varphi_{4k+2} - 20\varphi_{4k+3} + 3\varphi_{4k+4}\right], \qquad (42)$$

$$\varphi_{4k+9/4} = \frac{1}{2048}\left[35\varphi_{4k} - 252\varphi_{4k+1}1890\varphi_{4k+2} + 420\varphi_{4k+3} - 45\varphi_{4k+4}\right], \qquad (43)$$

4. Test Examples

In this section three examples are introduced to illustrate the effectiveness of the investigations.

4.1 Example: this example is linear with weakly singular kernel:

$$\varphi(t) = \frac{1}{\sqrt{t+1}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\left(\frac{1-t}{1+t}\right) - \frac{1}{4}\int_0^t (t-s)^{-\frac{1}{2}}\varphi(s)ds, \qquad (44)$$

This has an exact solution of the form:

$$\varphi(t) = \frac{1}{\sqrt{t+1}}. \qquad (45)$$

4.2 Example: this nonlinear example with weakly singularity:

$$\varphi(t) = \sqrt{t} + \frac{3\pi}{8}t^2 - \int_0^t (t-s)^{-\frac{1}{2}}\left[\varphi(s)\right]^3 ds, \qquad (46)$$

This has an exact solution of the form:

$$\varphi(t) = \sqrt{t}. \qquad (47)$$

4.3 Example: the last one is nonlinear and takes the form:

$$\varphi(t) = \sqrt{t}\left(1 - \frac{4}{3}t\right) + \int_0^t (t-s)^{-\frac{1}{2}}\left[\varphi(s)\right]^2 ds, \qquad (48)$$

with exact solution:

$$\varphi(t) = \sqrt{t}\,. \tag{49}$$

4.4 Results

Here the results of the pervious examples are presented. Tables (1) and (2) show the comparison between product Simpson's block by block and the modified Bool's block according to maximum error (E_m) and maximum relative error (RE_m) at the same (N).

Table (1) Results of Simpson's block by block

Ex.	N	β	E_m	RE_m
1	32 64	1.28 1.26	4.615×10^{-8} 3.933×10^{-9}	5.171×10^{-8} 4.521×10^{-9}
2	32 64	1.92 1.90	2.399×10^{-6} 2.210×10^{-7}	6.664×10^{-6} 6.365×10^{-7}
3	32 64	1.00 1.00	4.210×10^{-3} 3.766×10^{-4}	4.210×10^{-3} 3.766×10^{-4}

The following table shows the effect of graded nodes on product Bool's block by block, which reduces the maximum relative error.

Table (2) Results of the modified block by block

Ex.	N	β	E_m	RE_m
1	32 64	1.26 0.88	1.896×10^{-10} 1.109×10^{-10}	2.255×10^{-10} 1.121×10^{-10}
2	32 64	2.74 2.30	2.921×10^{-7} 2.156×10^{-8}	1.588×10^{-6} 6.037×10^{-7}
3	32 64	2.32 2.28	4.281×10^{-4} 1.252×10^{-6}	4.281×10^{-4} 1.399×10^{-6}

5. Conclusions

It's well known that the product block by block method is good self staring technique to solve weakly Volterra integral equations as Fredholm integral equations. In this paper two investigations are introduced to develop this method to solve nonlinear Volterra Integral Equation of the second kind with weakly singular kernel. Firstly we use Bool's quadrature rule which has high order of convergence. Secondly the Bool's block is adapted to use on graded nodes. The comparison between product Simpson's block and product Bool's block shows that the Maximum relative error of Bool's block becomes less than that of product Simpson's block.

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طريقة مضروبات الحزم عالية الرتبة لحل معادلة فولترا التكاملية غير الخطية من النوع الثاني ذات النواه المفردة

جمال عطية-حامد نور-جلال البغدادي

ملخص البحث. في هذا البحث تم تطوير طريقة مضروبات الحزم لحل معادلة فولترا التكاملية من النوع الثاني غير الخطية ذات النواة التي تحتوى على شذوذ ضعيف. وقد طورت طريقة مضروبات الحزم باقتراحين. اولا : فقد تم استخدام تربيعات باول كطريقة عالية الرتبة فى التقارب. وثانيا: فهو تهيئة الطريقة للاستخدام علي شبكة متدرجة.

وقد تم اختبار الطريقة وتم مقارنة النتائج بطريقة مضروبات حزم سيمسون ومقارنة الحل ايضا بالحل علي شبكة ذات مسافات متساوية وكانت النتائج هي الافضل.