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Using a Parametric Function to Study the Deformation of the Hyperbolic Secant Distribution

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Abstract. The main aim of this paper is to define and study the $q(w)$-deformed hyperbolic secant distribution. Here, we apply the deformation technique by introducing a parametric function $q(w)$ under some certain appropriate assumptions. Some important corresponding properties are discussed. Moreover, some corresponding available measures and functions of this distribution are derived. Finally, a particular case is presented and elucidated.

Keywords: Generalized hyperbolic functions, Generalized hyperbolic secant distribution, Hyperbolic secant distribution, q(t)-Hyperbolic function, q-DHS distribution.

AMS Subject Classification: 60 E05; 60 E10; 62 E17; 62 E20.

1. Introduction

Recently, the concept of deformation technique has been exploited to a great extent in several fields of sciences [1-4, 7-10, 17]. Other techniques were suggested and discussed in some literatures [6, 11-13, 18, 19]. The deformation technique has been applied for the hyperbolic and trigonometric functions. Specially, the author used this technique previously for the probability distributions, precisely for the hyperbolic secant distribution [8-10]. The q-deformed hyperbolic secant distribution, which is denoted by q-DHS distribution, has been constructed and its properties have been reviewed and discussed. This distribution has been obtained by introducing a positive deformation parameter "q" [9].

Our purpose here is to present and study an extension of *q*-DHS distribution by introducing a positive parametric function $q(w)$. Moreover, we use a linear function of the mentioned random variable with coefficients as functions of the scalar parameter *w*. For this study, we will consider some appropriate assumptions with respect to the introduced parametric function as well as the used coefficients in the mentioned function of the random variable.

This paper is organized as follows: Section 2 deals with the original hyperbolic secant distribution "HS distribution" and *q*-DHS distribution with some main interesting properties. The $q(w)$-deformed hyperbolic secant distribution " $q(w)$-DHS distribution" is established and moreover some corresponding definitions and properties are illustrated in section 3. Some corresponding functions and measures of the $q(w)$-DHS distribution are derived in section 4. In Section 5 using the Maximum Likelihood method "ML method" with respect to the constructed distribution to obtain on the Maximum Likelihood Estimation "MLE" of $q(w)$ is explained. Section 6 contains an illustrative example. Finally, the paper has been concluded in Section 7.

2. Definition of the *q*-Deformed Hyperbolic Secant Distribution

According to [5, 8-10, 12, 20], the pdf of HS distribution of the continuous random variable X is given by

$$f_{\text{HS}}(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi x}{2}\right); \qquad x \in \mathrm{R}. \tag{1}$$

This distribution is symmetric about zero with unit variance and it has some closed forms for some corresponding functions and measures.

The *q*-DHS distribution was originally introduced by the author [9]. It has probability density function "pdf"

$$f_{q-\mathrm{DHS}}(x\,;q)=\frac{\sqrt{q}}{2}\,\mathrm{sech}_q\left(\frac{\pi\,x}{2}\right); \qquad x\in\mathrm{R}, \tag{2}$$

where q is a real positive scalar parameter and

$$\mathrm{sech}_q\,x=\frac{1}{\cosh_q x}=\frac{2}{e^{x}+q\,e^{-x}}.$$

This is a family of continuous probability distributions in which the deformation parameter q can be used to introduce skew. Each *q*-DHS distribution is unimodal with unit variance. The corresponding moments-generating function "mgf", characteristic function "cf", cumulants-generating function "cgf" and score function for this distribution have been derived in closed forms which depend on q. Furthermore, all moments of *q*-DHS distribution exist and the mean, the median and the mode have equal non-zero values as a function of q. This family of *q*-DHS distributions was discussed in some details in [9].

3. The *q(w)*-Deformed Hyperbolic Secant Distribution

3.1 Definition of the *q(w)*-DHS distribution

Throughout this paper we consider the deformation technique for which a positive parametric function $q(w)$ is introduced as a factor of the exponential decay part of the hyperbolic secant function "HS function" [1]. The $q(w)$-DHS distribution is defined by means of the $q(w)$-deformation for HS functions. Firstly, we explain the concept and some properties of the deformed hyperbolic functions by introducing an arbitrary parametric function $q(w)$ and extending the random variable X by $\varphi=\varphi(X,w)$, where $.w\in R$

Definition 1. *Let* $q(w)$ *be an arbitrary real positive parametric differentiable function of* w *, deformed hyperbolic functions to be -*$q(w)$*. We define the* $w\in R$ *a family of the functions* $\sinh_{q(w)}\varphi, \cosh_{q(w)}\varphi, \tanh_{q(w)}\varphi, \mathrm{sech}_{q(w)}\varphi, \coth_{q(w)}\varphi$ *andas* $\mathrm{csch}_{q(w)}\varphi$

$$\sinh_{q(w)}\varphi = \frac{e^{\varphi} - q(w)e^{-\varphi}}{2},\ \cosh_{q(w)}\varphi = \frac{e^{\varphi} + q(w)e^{-\varphi}}{2},\ \tanh_{q(w)}\varphi = \frac{\sinh_{q(w)}\varphi}{\cosh_{q(w)}\varphi},$$
$$\coth_{q(w)}\varphi = \frac{\cosh_{q(w)}\varphi}{\sinh_{q(w)}\varphi},\quad \operatorname{sech}_{q(w)}\varphi = \frac{1}{\cosh_{q(w)}\varphi},\quad \operatorname{csch}_{q(w)}\varphi = \frac{1}{\sinh_{q(w)}\varphi}; \tag{3}$$

where $\varphi = \varphi(x,w)$ *is a real differentiable function of* x *and* w *, and it is also linear function in* x *with positive partial derivative with respect to* x *, i.e.* $\varphi = C(w)x + D(w)$, $C(w) \in (0, \infty)$ *as a derivative of* φ *with respect to* x *, and* $D(w) \in R$ *. The parametric function* $q(w)$ *is called the deformation parametric function.*

Lemma 1. *A family of* $q(w)$*-deformed hyperbolic functions satisfies the following relations of the first derivatives of* $\sinh_{q(w)}\varphi,\ \tanh_{q(w)}\varphi,\ \cosh_{q(w)}\varphi,\ \operatorname{sech}_{q(w)}\varphi$ *with respect to* x *:*

$$(\sinh_{q(w)}\varphi)' = C(w)\cosh_{q(w)}\varphi,\ (\tanh_{q(w)}\varphi)' = C(w)q(w)\operatorname{sech}^2_{q(w)}\varphi,$$
$$(\cosh_{q(w)}\varphi)' = C(w)\sinh_{q(w)}\varphi,\ (\operatorname{sech}_{q(w)}\varphi)' = -C(w)\operatorname{sech}_{q(w)}\varphi\ \tanh_{q(w)}\varphi. \tag{4}$$

Furthermore, if $q(w) \neq 1$ *then* $\sinh_{q(w)}\varphi$ *is not odd function with respect to* φ *and* $\cosh_{q(w)}\varphi$ *is not even function with respect to* φ*, i.e.*

$$\sinh_{q(w)}(-\varphi) = -q(w).\sinh_{\frac{1}{q(w)}}\varphi,\qquad \cosh_{q(w)}(-\varphi) = q(w).\cosh_{\frac{1}{q(w)}}\varphi.$$

Moreover, the following relations are satisfied:

$$\cosh^2_{q(w)}\varphi - \sinh^2_{q(w)}\varphi = q(w),\quad \tanh^2_{q(w)}\varphi = 1 - q(w).\operatorname{sech}^2_{q(w)}\varphi,$$

$$\coth^2_{q(w)}\varphi = q(w).\operatorname{csch}^2_{q(w)}\varphi + 1.$$

Proof: Based on [1, 9, 15] and Definition 1, we can directly prove this lemma.

The main idea of the suggested deformation technique is to generalize the HS-distribution in an alternative formula which depends on a real positive parametric function and also to study its important corresponding characteristics.

As an immediate consequence of previous definition and lemma, we can define the pdf of the constructed $q(w)$-DHS distribution as the following.

Definition 2. *Let* $X_{q(w)-\mathrm{DHS}}$ *be a continuous random variable. The variable* $X_{q(w)-\mathrm{DHS}}$ *has a* $q(w)$*-DHS distribution with a positive real parametric function* $q(w)$, *if its pdf given by*

$$f_{q(w)-\mathrm{DHS}}(\varphi; q(w)) = \frac{C(w)\sqrt{q(w)}}{2}\,\mathrm{sech}_{q(w)}\left(\frac{\pi\varphi}{2}\right);\ x, w \in \mathrm{R}, \tag{5}$$

where $q(w) \in (0,\infty)$ *and* $\varphi = \varphi(x, w) \in R$ *. In this case,* $X_{q(w)-\mathrm{DHS}}$ *is said to be a* $q(w)$*-DHS random variable with a parametric function* $q(w)$*, defined over* R *. Furthermore, the corresponding real valued cdf* $F_{q(w)-\mathrm{DHS}}(\varphi; q(w))$ *is*

$$F_{q(w)-\mathrm{DHS}}(\varphi; q(w)) = \frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{1}{\sqrt{q(w)}}\sinh_{q(w)}\left(\frac{\pi\varphi}{2}\right)\right], \tag{6}$$

with the inverse cdf (critical value)

$$x_{\alpha}^{q(w)-\mathrm{DHS}} = \frac{2}{\pi . C(w)}\left[\operatorname{arcsinh}\left[\tan\left(\pi\left(\frac{1}{2}-\alpha\right)\right)\right] - \frac{1}{2}\ln\frac{1}{q(w)}\right] - \frac{D(w)}{C(w)}, \tag{7}$$

where $P[X > x_{\alpha}^{q(w)-\mathrm{DHS}}] = 1 - F_{q(w)-\mathrm{DHS}}(x_{\alpha}^{q(w)-\mathrm{DHS}}) = \alpha,\ \alpha \in (0,1)$.

Without loss of generality, let $D(w) = 0$ and in this case the values $x_{\alpha}^{q(w)-\mathrm{DHS}}$ for some different values of w and for each fixed function $q(w)$ using (7) can be easily computed.

3.2 Properties of the *q(w)*-DHS distribution

Obviously, from the following figures (Figures 1 and 2), the exponential tail behavior of the $q(w)$-DHS distribution guarantees the existence of the expectation of the $q(w)$-DHS variable $X_{q(w)-\mathrm{DHS}}$ and generally all moments. In particular, the expectation of the variable $X_{q(w)-\mathrm{DHS}}$ and also the squared variable $X^2_{q(w)-\mathrm{DHS}}$ can be derived and given respectively by

$$\mu = E[X_{q(w)-\mathrm{DHS}}] = \frac{2}{\pi\, C(w)} \ln[\sqrt{q(w)}]\,,$$
$$E[X^2_{q(w)-\mathrm{DHS}}] = \frac{1}{C^2(w)} + \frac{4}{\pi^2\, C^2(w)} (\ln[\sqrt{q(w)}])^2. \tag{8}$$

This implies that the variance σ^2 equals $1/C^2(w)$.

We will next propose some properties of $q(w)$-DHS distribution through some possible propositions.

Proposition 1. *The* $q(w)$-DHS *distribution with a positive real valued parameter* $q(w)$ *is symmetric about 0 for* $q(w)=1$. *Moreover, it skewed more to the right for* $q(w)<1$ *and skewed more to the left for* $q(w)>1$. *For all positive real values of* $q(w)$, *the kurtosis is always constant.*

Different densities for $q(w)$-DHS distribution with $q(w)>1$ and their corresponding densities with $q(w)<1$ for some values of w and for each fixed parametric function are plotted in Figure 1. Moreover, Figure 2 illustrates $q(w)$ the pdf for $q(w)$-DHS distribution with $q(w)=1$

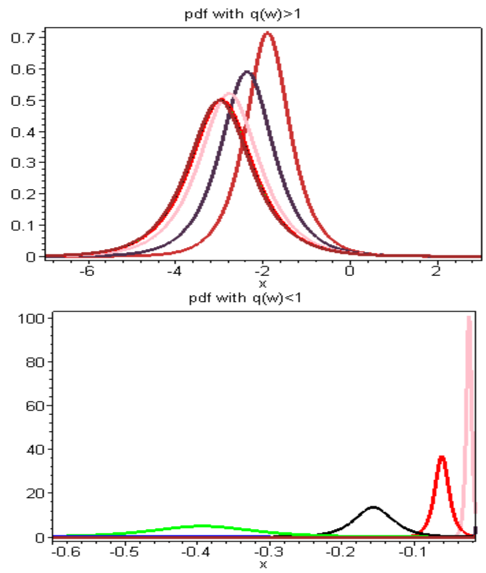

Figure 1: Probability density function for the $q(w)$-DHS distribution for different values of $q(w)$

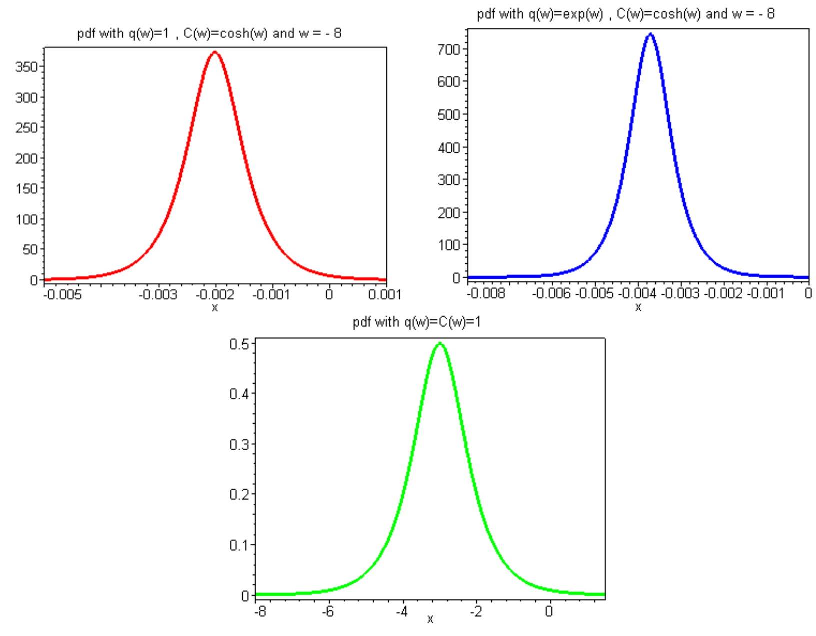

Figure 2: Probability density function for the $q(w)$-DHS distribution for the case $q(w)=1$

Graphically, the previous given statements in Proposition 1 are valid. Computation-ally, we can find that the density corresponding to (5) has larger (smaller) mean, when the value of $q(w)$ is increasing (decreasing).

Proposition 2. *The score function* $S_{q(w)-\mathrm{DHS}}(\varphi; q(w))$ *of the* $q(w)$-DHS *variable* $X_{q(w)-\mathrm{DHS}}$ *with* $q(w)>0$ *is given by*

$$S_{q(w)-\mathrm{DHS}}(\varphi; q(w)) = \frac{\pi}{2} C(w) \tanh_{q(w)} \frac{\pi\varphi}{2}. \tag{9}$$

Setting $q(w)=1$ and $C(w)=1$, *the last equation reduces to* $S_{\mathrm{HS}}(x) = \frac{\pi}{2}\tanh\frac{\pi x}{2}$*, where* $S_{\mathrm{HS}}(x)$ *is the score function of the HS distribution. Moreover, when* $q(w)=q$ *(i.e. parameter)* and $C(w)=1$, *the equation* (9) *reduces to* $S_{q-\mathrm{DHS}}(x) = \frac{\pi}{2}\tanh_q \frac{\pi x}{2}$ *which is the score function of the* q-DHS *variable* $X_{q-\mathrm{DHS}}$ *with* $q>0$.

Proof: Based on [16] and other literatures, the score function of a probability distribution is defined by $S(x) = -\frac{(\mathrm{pdf})'}{\mathrm{pdf}}$. By using (5), the form (9) of $S_{q(w)-\mathrm{DHS}}(\varphi; q(w))$ can be obtained with the reduced cases $S_{\mathrm{HS}}(x)$ and $S_{q-\mathrm{DHS}}(x) = \frac{\pi}{2}\tanh_q\frac{\pi x}{2}$ for $q(w)=1$, $C(w)=1$ and $q(w)=q$, $C(w)=1$, respectively.

The next proposition indicates to the unimodality of the $q(w)$-DHS distribution for all positive real values of the parametric function $q(w)$.

Proposition 3. *The* $q(w)$-DHS *distribution is unimodal for* $q(w)>0$.

Proof: Based on the function $f_{q(w)-\mathrm{DHS}}(\varphi; q(w))$ in (5), we aim to show this function is unimodal for all choices of $q(w)$ Since $f_{q(w)-\mathrm{DHS}}(\varphi; q(w))$ is a continuously differentiable function, the only critical points for this function satisfy $f'_{q(w)-\mathrm{DHS}}(\varphi; q(w))=0$ (the derivative with respect to x). Thus, we want to prove that the last equation has exactly one root, and that this yields a relative maximum. Since $\lim_{\varphi \to \pm\infty} f_{q(w)-\mathrm{DHS}}(\varphi; q(w)) = 0$, then if there is one critical point, it must yield the absolute maximum, so we need to prove there is exactly one root to the derivative equation. After simplification, this can be seen to be equivalent to proving $(\mathrm{sech}_{q(w)} \frac{\pi\,\varphi}{2}).(\tanh_{q(w)} \frac{\pi\,\varphi}{2}) = 0$ has exactly one root.

Set $\varphi(x; q(w)) = \frac{2}{\pi}(y + \ln[\sqrt{q(w)}\,])$, the last statement is equivalent to showing the equation $\mathrm{sech}(y)\tanh(y) = 0$ has exactly one root $y = 0$ in $\mathbb{R}$. This means that the equation $f'_{q(w)-\mathrm{DHS}}(\varphi; q(w)) = 0$ has only the root $\varphi^* = \varphi(x^*, w) = \frac{2}{\pi}\ln[\sqrt{q(w)}\,]$ (i.e. $x^* = \frac{2}{\pi . C(w)}\ln[\sqrt{q(w)}\,]$) in $\mathbb{R}$. Since $f''_{q(w)-\mathrm{DHS}}(\varphi^*; q(w)) < 0$ with $\varphi^* = \varphi(x^*, w)$, then the point x^* is the maximum value of the $q(w)$-DHS distribution. It then also follows this yields a relative maximum (and hence absolute maximum) since $f'_{q(w)-\mathrm{DHS}}(\varphi; q(w))$ is positive to the left of the root x^*, and negative to the right (Figure 3.).

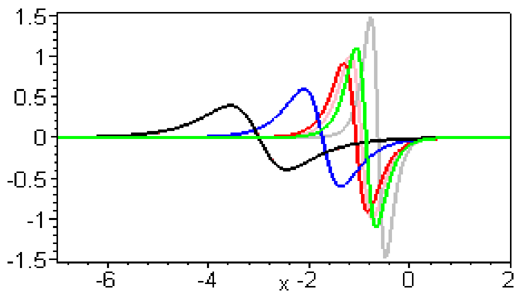

Figure 3: Derivative of the unimodal pdf of $q(w)$-DHS distributions with $q(w) > 0$

Note that, the mode for $q(w)$-DHS distribution have the above value of the root x^*, which equals the obtained mean.

Proposition 4. *The mode and the median for the $q(w)$-DHS distribution with $q(w) \in (0, \infty)$ have the same value of the mean.*

Proof: Due to the unimodality of the mentioned distribution, the previous obtained results and the fact that the median of the unimodal distribution lies between the mean and the mode of the same distribution, we can find that the given statement in the proposition is valid, i.e.

$$\mathrm{Mode}_{q(w)\text{-DHS}} = \mathrm{Median}_{q(w)\text{-DHS}} = \frac{2}{\pi\, C(w)} \ln[\sqrt{q(w)}\,], \quad q(w) > 0 \tag{10}$$

Note that, the case of $q(w) = q$ (where $q > 0$), $C(w) = 1$, the q-DHS distribution is recovered and also the case of $q(w) = 1$, $C(w) = 1$, gives the original HS distribution.

4. Moment-generating Function of the *q*(w)-DHS Distribution

In this section, we will derive the corresponding closed forms for the mgf, the cgf and also the cf of the $q(w)$-DHS distribution. Moreover, the moments of $X_{q(t)-\mathrm{DHS}}$ can be deduced from the mgf. Consequently, the corresponding skewness and kurtosis coefficients of the constructed $q(w)$-DHS distribution will be determined.

Proposition 5. *The* mgf *function* $\mathrm{M}_{q(w)-\mathrm{DHS}}(t;\, q(w))$ *of with* $X_{q(w)-\mathrm{DHS}}$ *is given by* $q(w) > 0$

$$\mathrm{M}_{q(w)-\mathrm{DHS}}(t; q(w)) = \mathrm{e}^{\frac{2t}{\pi . C(w)} \ln[\sqrt{q(w)}]} \sec t\,, \qquad |t| < \frac{\pi\, C(w)}{2} \,. \tag{11}$$

In particular, all moments of $X_{q(w)-\mathrm{DHS}}$ *exist.*

Proof: By using the substitutions $\varphi(x; q(w)) = \frac{2}{\pi}[y + \ln(\sqrt{q(w)})]$ and $B = \frac{2t}{\pi C(w)}$, we find that the mgf of the variable $X_{q(w)-\mathrm{DHS}}$ is given by

$$\mathrm{M}_{q(w)-\mathrm{DHS}}(t; q(w)) = \frac{1}{\pi} \mathrm{e}^{B \ln[\sqrt{q(w)}]} \int_{-\infty}^{\infty} \mathrm{e}^{B\, y} \operatorname{sech} y \, dy \,. \tag{12}$$

According to [14], we can find that the following integration, which can be easily worked out with the help of some mathematical packages

$$\int_{-\infty}^{\infty} e^{B\,y} \ \operatorname{sech} y \ dy = \pi \ \sec t \ , \quad |B| < 1 \qquad (13)$$

From (12) and (13), the closed form (11) of mgf of $X_{q(w)-\text{DHS}}$ can be obtained

Proposition 6. *The first four non-central moments of* $X_{q(w)-\text{DHS}}$ *with* $q(w) > 0$ *are given by*

$$\mu_1' = \frac{1}{\pi . C(w)} \ln[q(w)], \quad \mu_2' = \frac{1}{C^2(w)} + \frac{1}{\pi^2 C^2(w)} (\ln[q(w)])^2 \ ,$$

$$\mu_3' = \frac{3}{\pi . C^3(w)} \ln[q(w)] + \frac{1}{\pi^3 C^3(w)} (\ln[q(w)])^3 \ ,$$

$$\mu_4' = \frac{5}{C^4(w)} + \frac{6}{\pi^2 C^4(w)} (\ln[q(w)])^2 + \frac{1}{\pi^4 C^4(w)} (\ln[q(w)])^4 \ .$$

Note that the previous forms in Proposition 6 can be easily worked out with the help of some mathematical packages. In this case, the first four central moments of $X_{q(w)-\text{DSH}}$ are

$$\mu_1 = 0 \ , \ \mu_2 = \frac{1}{C^2(w)} \ , \ \mu_3 = 0 \ , \ \mu_4 = \frac{5}{C^4(w)} \ .$$

This implies that the skewness and the excess kurtosis are $\gamma = 0$ and $\beta = 2$ respectively.

Using the relation between the cf and the mgf, we can obtain the cf of the $q(w)$-DHS distribution in the following closed form:

$$\Psi_{q(w)-\text{DHS}}(t;\, q(w)) = e^{\frac{2\,i\,t}{\pi . C(w)} \ln[\sqrt{q(w)}]} \ \operatorname{sech} t \ , \qquad |t| < \frac{\pi C(w)}{2}. \qquad (14)$$

The next proposition gives the closed form of the cgf and the used closed form to calculate the r-*th* cumulant k_r of the $q(w)$-DHS distribution.

Proposition 7. *Assume that the variable* $X_{q(w)-\mathrm{DHS}}$ *follows the* $q(w)$-DHS *distribution with* $q(w) > 0$. *The corresponding cgf of this variable is given by*

$$\mathrm{K}_{q(w)-\mathrm{DHS}}(t;\, q(w)) = \frac{2t}{\pi C(w)} \ln[\sqrt{q(w)}\,] + \ln[\sec t\,],\ \ |t| < \frac{\pi C(w)}{2}, \tag{15}$$

and the r-th cumulant k_r, $r = 1, 2, 3, \ldots$, *of* $X_{q(w)-\mathrm{DHS}}$ *is determined by*

$$k_r = [\frac{d^r}{dt^r} \mathrm{K}_{q(w)-\mathrm{DHS}}(t;\, q(w))]_{t=0} = [\mathrm{K}^{(r)}_{q(w)-\mathrm{DHS}}(t;\, q(w))]_{t=0},\ \ r = 1, 2, 3, \ldots, \tag{16}$$

where

$$\mathrm{K}^{(1)}_{q(w)-\mathrm{DHS}}(t;\, q(w)) = \frac{2}{\pi . C(w)} \ln[\sqrt{q(w)}\,] + \tan t\ ,$$

$$\mathrm{K}^{(2)}_{q(w)-\mathrm{DHS}}(t;\, q(w)) = 1 + \tan^2 t\ ,$$

$$\mathrm{K}^{(3)}_{q(w)-\mathrm{DHS}}(t;\, q(w)) = 2 \tan t\ (1 + \tan^2 t)\ ,$$

$$\mathrm{K}^{(4)}_{q(w)-\mathrm{DHS}}(t;\, q(w)) = 2(1 + \tan^2 t)^2 + 4 \tan^2 t\ (1 + \tan^2 t)\ ,$$

$$\mathrm{K}^{(5)}_{q(w)-\mathrm{DHS}}(t;\, q(w)) = 16(1 + \tan^2 t)^2 \tan t + 4 \tan^3 t\ (1 + \tan^2 t)\ ,\ldots.$$

The r-*th* cumulants k_r of $X_{q(w)-\mathrm{DHS}}$ for some values of r can be easily worked out with the help of some mathematical packages and they have been calculated.

Moreover, the moments of $X_{q(w)-\mathrm{DHS}}$ are related with the cumulants, i.e.

$$\mu_1' = \mu = k_1,\ \sigma^2 = \mu_2 = \frac{1}{C^2(w)} k_2,\ \mu_3 = k_3,\ \mu_4 = \frac{1}{C^4(w)}[k_4 + 3(k_2)^2]\ ,\ldots,$$

and so on.

In the next section, we will illustrate the ML Method to determine a certain value of the parametric function that maximizes the probability (likelihood) of the sample data from the $q(w)$-DHS distribution.

5. Maximum Likelihood Parameter Estimation

To obtain the MLE for the parameter $q(w)$ for the $q(w)$-DHS distribution, start with the pdf of the $q(w)$-DHS distribution which is given in (5).

Suppose that $X_1, X_2, \ldots, X_n$ are an iid random sample from a $q(w)$-DHS distribution. Then the likelihood function is given by

$$L(x_1, x_2, \ldots, x_n \,|\, q(w)) = C^n(w)\, q^{n/2}(w) \prod_{i=1}^{n} [\exp(\frac{\pi\varphi_i}{2}) + q(w)\exp(-\frac{\pi\varphi_i}{2})]^{-1}, \quad (17)$$

with $\varphi_i = \varphi(x_i; w)$. The log-likelihood function is

$$\ell(w) = \ell(q(w)) = \frac{n}{2}\ln(C^2(w).q(w)) - \sum_{i=1}^{n} \ln[\exp(\frac{\pi\varphi_i}{2}) + q(w)\exp(-\frac{\pi\varphi_i}{2})]. \quad (18)$$

Taking the derivative of the log-likelihood function with respect to w and setting it equals zero yields

$$\frac{C(w)q(w)}{2C'(w)q(w) + C(w)q'(w)} \sum_{i=1}^{n} [\pi\varphi_i' \tanh_{q(w)}(\frac{\pi\varphi_i}{2}) + q'(w)\exp(-\frac{\pi\varphi_i}{2})\operatorname{sech}_{q(w)}(\frac{\pi\varphi_i}{2})] = n, \quad (19)$$

with $\varphi_i' = C'(w)x_i + D'(w)$. Solving (19) iteratively, then the MLE $\hat{q}(w) = q(\hat{w})$ can be obtained.

6. Application

Here we give an example as a particular case of the mentioned deformed distribution and explain most of the obtained results for this case.

Let $X_{q(w)-\text{DHS}}$ be a continuous random variable which follows the $q(w)$-DHS distribution with the parametric function $q(w) = \exp(w)$. We consider

. In this case we can easily find that the pdf of $\cosh(w)x+3 \quad \varphi=\varphi(x,w)$ given by, $X_{\exp(w)-\mathrm{DHS}}$

$$f_{\exp(w)-\mathrm{DHS}}(\varphi;\exp(w))=\frac{\exp(w/2)}{2\,\mathrm{sech}(w)}\,\mathrm{sech}_{\exp(w)}(\frac{\pi\,\varphi}{2});\ x,w\in\mathrm{R},$$

and the cdf of $X_{\exp(w)-\mathrm{DHS}}$ is

$$F_{\exp(w)-\mathrm{DHS}}(\varphi;\exp(w))=\frac{1}{2}+\frac{1}{\pi}\arctan[\exp(-w/2)\,\sinh_{\exp(w)}(\frac{\pi\,\varphi}{2})],$$

with the critical value

$$x_{\alpha}^{\exp(w)-\mathrm{DHS}}=\mathrm{sech}(w)\ .[\frac{2}{\pi}\ (\mathrm{arc}\sinh[\tan(\pi(\frac{1}{2}-\alpha))\ )+\frac{w}{2}\ )-3],$$

where $P[X>x_{\alpha}^{\exp(w)-\mathrm{DHS}}]=1-F_{\exp(w)-\mathrm{DHS}}(x_{\alpha}^{\exp(w)-\mathrm{DHS}})=\alpha,\ \alpha\in(0,1)$.

We can find that, the expectation of the $\exp(w)$-DHS variable $X_{\exp(w)-\mathrm{DHS}}$ (the mode, the median and the 1st non-central moment) and also the squared variable $X^{2}_{\exp(w)-\mathrm{DHS}}$ (the 2nd non-central moment) are given respectively by $\mu_1'=w\ \mathrm{sech}(w)\ /\pi$ and $\mu_2'=[1+w^{2}/\pi^{2}]\mathrm{sech}^{2}(w)$. Moreover, the variance of this variable is $\sigma^{2}=\mathrm{sech}^{2}(w)$. Also, The mgf function of the $X_{\exp(w)-\mathrm{DHS}}$ is given by

$$\mathrm{M}_{\exp(w)-\mathrm{DHS}}(t;\exp(w))=\sec(t)\ \exp[\frac{t\ w\ \ \mathrm{sech}(w)}{\pi}];\ |t|<\frac{\pi}{2\ \mathrm{sech(w)}}.$$

The 3rd and 4th non-central moments of the variable $X_{\exp(w)-\mathrm{DHS}}$ can be obtained as

$$\mu_3'=[3w/\pi+w^{3}/\pi^{3}]\mathrm{sech}^{3}(w) \quad \text{and} \quad \mu_4'=[5+6w^{2}/\pi^{2}+w^{4}/\pi^{4}]\,\mathrm{sech}^{4}(w).$$

Thus, the first four central moments of the variable $X_{q(w)-\text{DSH}}$ can be also derived and the results are $\mu_1 = 0$, $\mu_2 = \sigma^2 = \text{sech}^2(w)$, $\mu_3 = 0$, $\mu_4 = 5.\text{sech}^4(w)$. Then we find that $\gamma = 0$ and $\beta = 2$. According to the forms (14) and (15), the cf and cgf of the $\exp(w)$-DHS distribution are given respectively in the following forms:

$$\Psi_{\exp(w)-\text{DHS}}(t;\ \exp(w)) = \text{sech}(t)\ \exp[\frac{i\,t.w\ \ \text{sech}(w)}{\pi}],$$

$$\text{K}_{\exp(w)-\text{DHS}}(t;\ \exp(w)) = \ln[\sec(t)] + \frac{t.w\ \ \text{sech}(w)}{\pi},$$

with $|\,t\,| < \dfrac{\pi}{2\ \text{sech}(w)}$. By using the last form of the cgf , we can find that:

$$\text{K}^{(1)}_{\exp(w)-\text{DHS}}(t;\ \exp(w)) = \frac{w\ \ \text{sech}(w)}{\pi} + \tan(t),$$

and $\text{K}^{(k)}_{\exp(w)-\text{DHS}}(t;\ \exp(w))$, $k = 2,3,4,...,$ as in Proposition 7. Moreover, the moments of $X_{q(w)-\text{DHS}}$ are related with the cumulants,

$\mu_1' = \mu = k_1,\ \sigma^2 = \mu_2 = \text{sech}^2(w).k_2,\ \mu_3 = k_3,$

$\mu_4 = \text{sech}^4(w)\ [k_4 + 3(k_2)^2]$, ..., and so on.

Finally, by solving the following nonlinear system in w iteratively,

$$\frac{1}{2\tanh(w)+1}\sum_{i=1}^{n}[\pi.x_i\ .\sinh(w)\ \tanh_{\exp(w)}(\frac{\pi\,\varphi_i}{2}) + \exp(-\frac{\pi\,\varphi_i}{2} + w)\,\text{sech}_{\exp(w)}(\frac{\pi\,\varphi_i}{2})] = n,$$

with $\varphi_i = \cosh(w)\ x_i + 3$ and $\varphi_i' = \sinh(w).x_i$, $i = 1,2,...,n$, one can obtain $\hat{w}$ and thus the MLE of $q(w) = \cosh(w)$ is $\hat{q}(w) = \cosh(\hat{w})$.

Different densities for the $\exp(w)$-DHS distribution with $\exp(w) > 1$ and their corresponding densities with $\exp(w) < 1$ for some values of w are plotted in Figure 1. Moreover, Figure 2 illustrates the pdf for the $\exp(w)$-DHS distribution with $\exp(w)=1$. The derivative of the unimodal pdf of $\exp(w)$-DHS distributions is explained in Figure 3.

7. Conclusions

This paper discussed the construction of a family of the $q(w)$-DHS distributions which can be considered as a corresponding extension of a family of the q -DHS distributions. We defined the $q(w)$-deformed hyperbolic functions which have been implemented by applying a deformation by introducing a positive real valued parametric function. We studied the effect of this introduced parametric $q(w)$ function in compare with the previous studies on the hyperbolic secant distribution. Here, we introduced the deformation parametric function $q(w)$ as a factor of the exponential decay part of the HS distribution. Moreover, we considered a differentiable real valued function $\varphi = \varphi(X, w)$ instead of the random variable with positive partial x . We assumed that this function is linear function in X derivative with respect to x. We found that the constructed family of the $q(w)$-DHS distributions is unimodal. In general, it has variance with value not equals 1. We noted also that the derived closed forms of the corresponding mgf, cf, cgf and score function for the $q(w)$-DHS distributions depend on $q(w)$ and the derivative $C(w)$ of φ. Furthermore, some main properties of this constructed family of deformed distributions were discussed and we have noted that their moments exist. Moreover, there is unique value of their mean, median and mode which still also as a function of the deformation parametric $q(w)$ and $C(w)$. The skewness and excess kurtosis of these constructed distributions are still respectively equal to 0 and 2. The ML method to determine the MLE for the parameter $q(w)$ has been illustrated and we obtained a nonlinear system which can be solved iteratively but by using high processing systems of computers. A particular case of the $q(w)$-DHS distributions has been presented.

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استخدام دالة بارامترية لدراسة تشوه توزيع القاطع الزائدي

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ملخص البحث. في هذا البحث نقدم دراسة تأثير تطبيق طريقة التشوه من خلال إدخال دالة بارامترية بفروض ملائمة على توزيع القاطع الزائدي الاحتمالي ونناقش بعض الخصائص المصاحبة الهامة . هذا بالإضافة لإستخلاص بعض الصيغ الرياضية لبعض الدوال والمقاييس المرتبطة بالتوزيع الاحتمالي المتولد نتيجة التشوه الحادث.