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Analytical Solutions of Some Nonlinear Space-Time Fractional Differential Equations via Improved Exp-Function Method

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Abstract. In this paper, the improved exp-function method is proposed to solve fractional differential equations. The method is applied to solve space-time fractional Kaup–Kupershmidt equation, space-time fractional shallow water equation, and space-time fractional Boussinesq equation. Among these solutions, some are found for the first time.

Keywords: *Improved exp-function method, nonlinear fractional differential equation, modified Riemann. Liouville derivative, exact solution.*

1. Introduction

Nonlinear partial differential equations (NLPDEs) of integer order are widely used as models to describe complex physical phenomena in various field of science such as fluid mechanics, plasma physics, optical fibers, biology, solid-state physics, chemical kinematics, and chemical physics. In the research of the theory of NLPDEs, searching for more explicit exact solutions to NLPDEs is one of the most fundamental and significant studies in recent years. With the help of computerized symbolic computation, much work has focused on the various extensions and applications of the known algebraic methods to construct the solutions to NLPDEs. There has been a variety of powerful methods. For example, these methods include the sine-cosine function method, tanh function method, projective Ricatti method, (G'/G)-expansion method, Lucas Ricatti method and Jacobi elliptic function method [1- 9]. He and Wu [10] proposed a straightforward and concise method, called exp-function method [11- 16], to obtain generalized solitary wave solutions of nonlinear PDEs.

Fractional differential equations (FDEs) involving fractional derivatives are the generalization of the classical differential equations of integer order. Fractional derivatives are useful in describing the memory and hereditary properties materials and processes. FDEs are widely used as models to express much important natural science such as chemistry, biology, mathematics, communication and particularly in almost all branches of physics. There are different kinds of fractional integration and differentiation operators. Searching for numerical and analytical solutions of FDEs has been a subject of intense study in recent years [17- 20]. There are different kinds of fractional integration and differentiation operators. The most famous one is the Riemann-Liouville definition [21], which has been used in various fields of science and engineering successfully, but this definition leads to the result that constant function differentiation is not zero. Caputo put definitions which give zero value for fractional differentiation of constant function, but these definitions require that the function should be smooth and differentiable [17- 18]. Recently, Jumarie derived definitions for the fractional integral and derivative called modified Riemann-Liouville [22- 25], which are suitable for continuous and non-differentiable functions and give differentiation of a constant function equal to zero. The modified Riemann-Liouville fractional definitions are used effectively in many different problems [26- 30]. In the literature, there are many effective methods to treat FDEs such as the Adomian decomposition method [31, 32], the variational iteration method [33], the homotopy perturbation method, the differential transform method, the finite difference method, the finite element method, the exponential function method [34, 35], the fractional sub-equation method [36- 39], the (G'/G)-expansion method [40, 41] and the first integral method [42]. Based on Jumarie's modified Riemann-Liouville derivative and the fractional Riccati equation $D_x^{\alpha}F(x)=\sigma+F(x)^2$, Zhang and Zhang in [36] introduced the sub-equation method for solving nonlinear time fractional biological population model and (4+1)-

dimensional space-time fractional Fokas equation. Guo et al [37] improved the sub-equation method, they obtained the analytical solutions of the space-time fractional Whitham-Broer-Kaup and generalized Hirota-Satsuma coupled KdV equations by introducing a new general ansätz. By extending the fractional Riccati equation [36] to the more general form $D_x^{\alpha} F(x) = A + B F(x)^2$, Abdel-Salam and Yousif [28] presented the fractional Riccati expansion method to obtain exact solutions of the space-time fractional Korteweg-de Vries equation, the space-time fractional RLW equation, the space-time fractional Boussinesq equation, and the space-time fractional Klein-Gordon equation. In addition, Li et al [43] extended fractional Riccati expansion method for solving the time fractional Burgers equation and the space-time fractional Cahn-Hilliard equation. Abdel-Salam et al [29] generalized the fractional Riccati expansion method to solve fractional differential equations with variable coefficients. Recently, Abdel-Salam and Al-Muhiameed [44] introduced the fractional mapping method by solving the fractional elliptic equation $D_x^{\alpha} F(x) = \sqrt{A + B F(x)^2 + C F(x)^4}$ and studied the space-time fractional combined KdV-mKdV equation. In addition, Abdel-Salam and Jazmati introduced the triple fractional Riccati expansion method to solve the nonlinear FDEs [45]. In this paper, the improved exp-function method were used to solve FDEs. The analytic solutions of the space-time fractional Kaup–Kupershmidt equation, the space-time fractional shallow water equation, and the space-time fractional Boussinesq equation are obtained.

The structure of this paper is as follows: some basic definitions of the fractional calculus and the description of the improved exp-function method introduced in section 2. In section 3, space-time fractional Kaup–Kupershmidt equation, space-time fractional shallow water equation, and space-time fractional Boussinesq equation are studied. In the last section, some conclusions are given.

2. Description of the method

In this section we present the improved exp-method to construct exact analytical solutions of nonlinear FDEs with the modified Riemann–Liouville derivative defined by Jumarie [22- 25]

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, & 0 < \alpha < 1 \\ [f^{(\alpha-n)}(x)]^{(n)}, & n \le \alpha < n+1,\ n \ge 1, \end{cases} \tag{1}$$

which has merits over the original one, for example, the α -order derivative of a constant is zero. Some properties of the Jumarie's modified Riemann–Liouville derivative are

$$D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \qquad \gamma > 0, \tag{2}$$

$$D_x^{\alpha}(c\, f(x)) = c\, D_x^{\alpha}\, f(x), \tag{3}$$

$$D_x^{\alpha}[f(x)g(x)] = g(x)D_x^{\alpha} f(x) + f(x)D_x^{\alpha} g(x), \tag{4}$$

$$D_x^{\alpha} f[g(x)] = f_g^{'}[g(x)]D_x^{\alpha} g(x), \tag{5}$$

$$D_x^{\alpha} f[g(x)] = D_g^{\alpha} f[g(x)](g_x^{'})^{\alpha}, \tag{6}$$

where c is constant. The formulas 4 - 6 follow from the fractional Leibniz rule and the fractional Barrow's formula. That is direct results of the equality $D_x^{\alpha} f(x) \cong \Gamma(\alpha+1) D_x f(x)$, which holds for non-differentiable functions. We present the main steps of this method as follows:

Suppose that the nonlinear FDE, say in two variables x and t, is given by:

$$P(u,\, D_t^{\alpha}u, D_x^{\alpha}u,\, D_t^{2\alpha}u,\, D_x^{2\alpha}u,\, \ldots) = 0, \qquad 0 < \alpha \le 1, \tag{7}$$

where $D_t^{\alpha}u,\, D_x^{\alpha}u$ and $D_y^{\alpha}u$ are Jumarie's modified Riemann–Liouville derivatives of u, $u = u(x,\, t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, other wise , a suitable transformation can transform equation (7) into such equation. The exp-function method for single-wave solution depend on the assumption that equation (7) has solution in the form

$$u(x,t) = u(\xi) = \frac{\sum_{i=0}^{p} a_i e^{i\xi}}{\sum_{j=0}^{q} b_j e^{j\xi}}, \qquad \xi = \frac{k\, x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \tag{8}$$

where $k,\, \omega,\, a_i$ and b_j are unknown constants to be determined and p, q are positive integers that could be freely chosen.

Substituting equation (8) into the FDE (7), the left-hand side of equation (7) converted into a polynomial in exp-function. Equating each coefficient of the exp-function to zero gives system of algebraic equations. Solving the set of equations, we can obtain the exact solution of equation (7).

3. Applications

In order to illustrate the effectiveness of the method, examples of mathematical and physical interests are chosen as follows:

3.1. The Kaup–Kupershmidt equation

The corresponding equation is the space-time fractional Kaup–Kupershmidt equation, for the internal solitary waves in shallow seas and atmosphere dust-acoustic solitary waves and ion acoustic waves in plasmas with negative ions,

$$D_t^{\alpha}u = D_x^{5\alpha}u - 20u\,D_x^{3\alpha}u - 50D_x^{\alpha}\,u\,D_x^{2\alpha}u + 80u^2 D_x^{\alpha}u, \qquad 0<\alpha\le 1, \tag{9}$$

which is a transformed generalization of the Kaup–Kupershmidt equation

$$u_t = u_{xxxxx} - 20u\,u_{xxx} - 50u_x\,u_{xx} + 80u^2 u_x. \tag{10}$$

That is the nonlinear fifth-order partial differential equation; it is the first equation in a hierarchy of integrable equations with Lax operator $\partial_x^3 + 2u\partial_x + u_x$. It has properties similar (but not identical) to those of the better-known KdV hierarchy in which the Lax operator has order two. In order to solve equation (9) by the improved exp-function method, we use the traveling wave transformation $u(x,\,t) = u(\xi)$, $\xi = \frac{k\,x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}$, based on this transformation for the terms in (9) containing fractional derivative, such as $D_t^{\alpha}u,\ D_x^{\alpha}u,\ D_x^{2\alpha}u,\ldots$, using (3) and (5) where $u(x,\,t)$ is a smooth and differentiable function, one can obtain that

$$\begin{aligned} &D_t^{\alpha}u(\xi) = u'\,D_t^{\alpha}\xi = \omega u', \\ &D_x^{\alpha}u(\xi) = u'\,D_x^{\alpha}\xi = k\,u', \\ &D_x^{2\alpha}u = D_x^{\alpha}(D_x^{\alpha}u(\xi)) = D_x^{\alpha}(k\,u') = k\,u''\,D_x^{\alpha}\xi = k^2u'',\ldots, \end{aligned} \tag{11}$$

then, equation (9) is reduced to the following nonlinear FODE:

$$\omega u' = k^5 u^{(5)} - 20u\,u''' - 50u'u'' + 80u^2u', \tag{12}$$

where primes denote derivatives with respect to ξ. Now we study the following cases:

case 1: $p=2,\ q=2$:

According to the improved exp-function method, the solution of equation (12) in this case can be written as:

$$u(\xi)=\frac{a_0+a_1 e^{\xi}+a_2 e^{2\xi}}{b_0+b_1 e^{\xi}+b_2 e^{2\xi}}, \qquad \xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \tag{13}$$

Substituting (13) into equation (12), equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations. Solving the resultant algebraic system for the unknowns $k,\ \omega,\ a_0, a_1, a_2, b_0, b_1$ and b_2, we obtain the solution sets

$$k=1,\ \ \omega=11,\ \ a_0=\frac{1}{2},\ a_1=-5,\ \ a_2=\frac{1}{2},\ b_0=1,\ \ b_1=2,\ \ \ b_2=1. \tag{14}$$

$$k=1,\ \ \omega=\frac{1}{16},\ \ a_0=\frac{1}{16},\ a_1=-\frac{5}{8},\ \ a_2=\frac{1}{16},\ b_0=1,\ b_1=2,\ b_2=1. \tag{15}$$

Thus, the solutions of the space-time fractional Kaup–Kupershmidt equation take the form

$$u_1=\frac{1-10e^{\xi}+e^{2\xi}}{2(1+2e^{\xi}+e^{2\xi})}, \qquad \xi=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{11t^{\alpha}}{\Gamma(1+\alpha)}, \tag{16}$$

$$u_2=\frac{1-10e^{\xi}+e^{2\xi}}{16(1+2e^{\xi}+e^{2\xi})}, \qquad \xi=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{\alpha}}{16\Gamma(1+\alpha)}, \tag{17}$$

To understand the effect of the fractional order α, we graph equation (16) with different value of α. Figure 1 shows the solution (16) in 3-dimension when the values of $\alpha=0.25,\ 0.5,\ 0.75, 1$. It has observed that the amplitude of the wave increased as the values of the fractional order derivative increase.

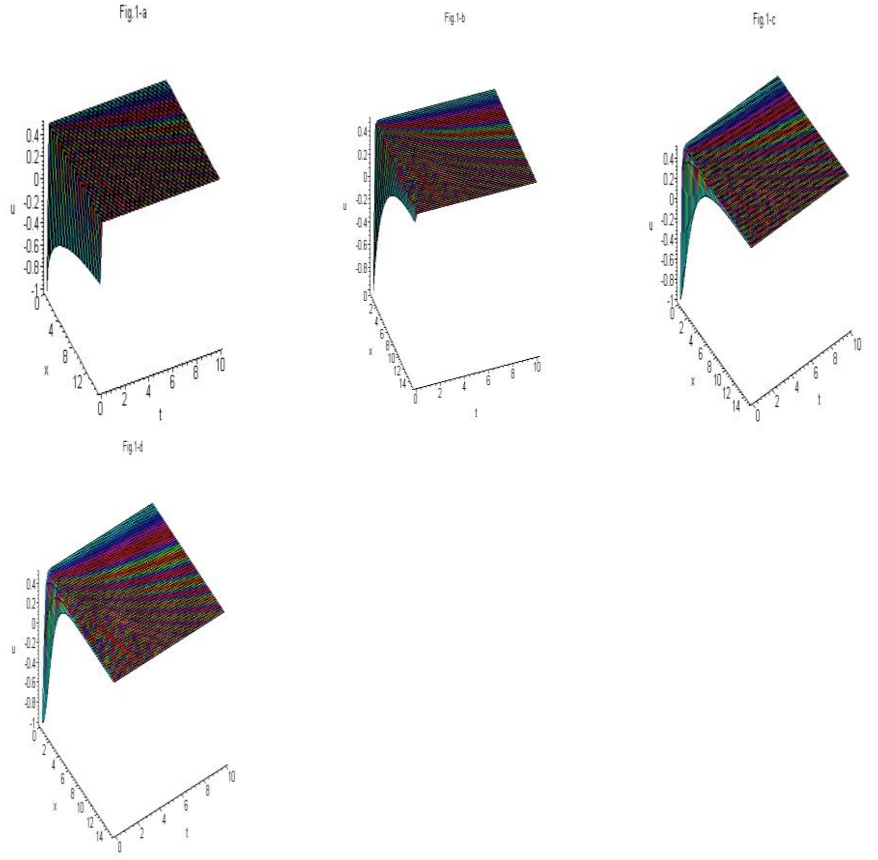

Figure 1: Evolutional behavior of u_1 with: (a) $\alpha = 0.25$; (b) $\alpha = 0.5$; (c) $\alpha = 0.75$; (d) $\alpha = 1$.

case 2: $p = 3,\ q = 3$:

According to the improved exp-function method, the solution of equation (12) in this case can be written as:

$$u(\xi) = \frac{a_0 + a_1 e^{\xi} + a_2 e^{2\xi} + a_3 e^{3\xi}}{b_0 + b_1 e^{\xi} + b_2 e^{2\xi} + b_3 e^{3\xi}}, \qquad \xi = \frac{k\,x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \tag{18}$$

Substituting (13) into equation (12), equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations. Solving the resultant algebraic system for the unknowns $k,\ \omega,\ a_0, a_1, a_2, a_3, b_0, b_1, b_2$ and b_3, we obtain the solution set

$$k=1,\quad \omega=11,\quad a_0=b_0=0,\quad a_1=\frac{b_2^2}{8},\quad a_2=-\frac{5b_2}{2},\quad a_3=\frac{1}{2},\quad b_1=\frac{b_2^2}{4},\quad b_3=1. \tag{19}$$

$$k=1,\quad \omega=\frac{1}{16},\quad a_0=b_0=0,\quad a_1=\frac{b_2^2}{64},\quad a_2=-\frac{5b_2}{16},\quad a_3=\frac{1}{16},\quad b_1=\frac{b_2^2}{4},\quad b_3=1. \tag{20}$$

Thus, the solutions of the space-time fractional Kaup–Kupershmidt equation take the form

$$u_3=\frac{b_2^2e^{\xi}-20b_2\,e^{2\xi}+4e^{3\xi}}{2(b_2^2e^{\xi}+4b_2\,e^{2\xi}+4e^{3\xi})},\qquad \xi=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{11t^{\alpha}}{\Gamma(1+\alpha)}, \tag{16}$$

$$u_4=\frac{b_2^2e^{\xi}-20b_2\,e^{2\xi}+4e^{3\xi}}{16(b_2^2e^{\xi}+4b_2\,e^{2\xi}+4e^{3\xi})},\qquad \xi=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{\alpha}}{16\,\Gamma(1+\alpha)}. \tag{17}$$

When $\alpha=1$, then the results are similar to those obtained by El-Sabbagh et al [64]

3.2 The generalized shallow water equation

Consider the generalized shallow water equation

$$u_{xxxt}+r\,u_x\,u_{xt}+s\,u_t u_{xx}-u_{xt}-u_{xx}=0, \tag{18}$$

where r and s are arbitrary nonzero constants. The shallow water wave equations describe the evolution of incompressible flow, neglecting density change along the depth. The shallow water wave equations are applicable to cases where the horizontal scale of the flow is much bigger than the depth of fluid. The shallow water equations have been extensively used for a wide variety of coastal phenomena, such as tide-currents, pollutant- dispersion storm-surges, tsunami-wave propagation**.** The space-time fractional shallow water equation, which is a transformed generalization of the shallow water equation, is defined as follows:

$$D_x^{3\alpha}D_t^{\alpha}u+rD_x^{\alpha}uD_x^{\alpha}D_t^{\alpha}u+sD_t^{\alpha}uD_x^{2\alpha}u-D_x^{\alpha}D_t^{\alpha}u-D_x^{2\alpha}u=0,\qquad 0<\alpha\le 1, \tag{19}$$

where $u=u(x,t)$, r , s are arbitrary constants and α is the fractional order derivative. In order to solve equation (15) by the improved exp-function method, we use the traveling wave transformation $u(x,\,t)=u(\xi)$, $\xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}$, based on this transformation, equation (9) is reduced to the following nonlinear FODE:

$$\omega k^3u^{(4)}+\omega k^2(r+s)u'u''-\omega ku''-k^2u''=0, \tag{20}$$

where the primes denote derivatives with respect to ξ. Now we study the following cases:

case 1: $p = 2,\ q = 2$:

According to the improved exp-function method, the solution of equation (20) in this case can be written as:

$$u(\xi) = \frac{a_0 + a_1 e^{\xi} + a_2 e^{2\xi}}{b_0 + b_1 e^{\xi} + b_2 e^{2\xi}}, \qquad \xi = \frac{k\, x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \tag{21}$$

Substituting (21) into equation (20), equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations. Solving the resultant algebraic system for the unknowns $k,\ \omega,\ a_0, a_1, a_2, b_0, b_1$ and b_2, we obtain the solution set

$$k = 1,\ \omega = \frac{1}{3},\quad a_1 = b_1 = 0,\ \ a_0 = \frac{b_0[-24 + a_2(r+s)]}{(r+s)}, \quad b_2 = 1. \tag{22}$$

Thus, the solutions of the space-time fractional shallow water equation take the form

$$u_{11}(\xi) = \frac{-24b_0 + a_2(r+s)[b_0 + e^{2\xi}]}{(r+s)[b_0 + e^{2\xi}]}, \qquad \xi = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{\alpha}}{3\Gamma(1+\alpha)}, \tag{23}$$

To understand the effect of the fractional order α, we graph equation (23) with different value of α . Figure 2 shows the solution (23) in 3-dimension when the values of $\alpha = 0.25,\ 0.5,\ 0.75, 1$, with selection of parameters $b_0 = 1, a_2 = 2,\ r = s = 1$. It has observed that the amplitude of the wave increased as the values of the fractional order derivative increase.

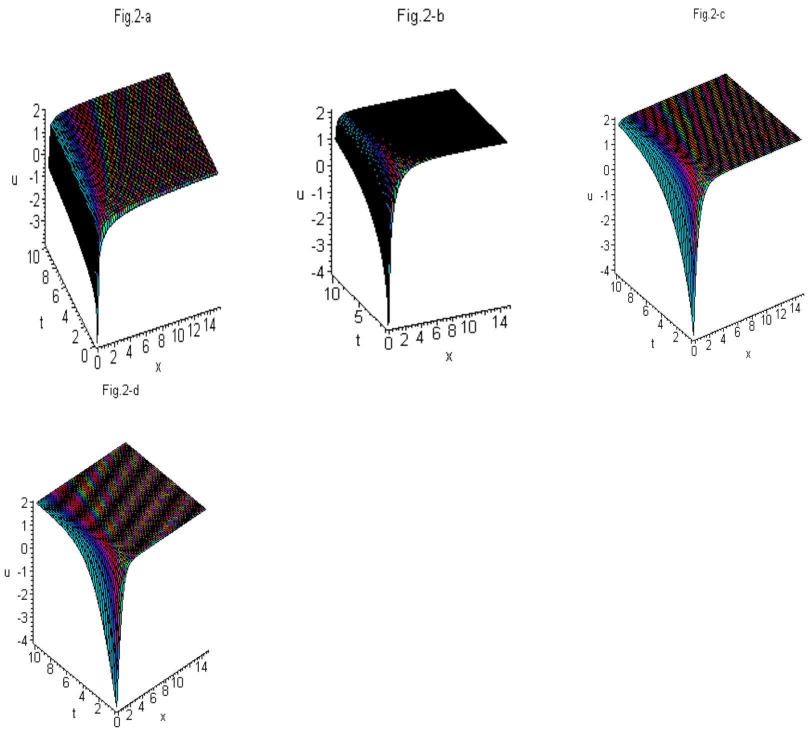

Figure 2: Evolutional behavior of u_1 with $b_0 = 1, a_2 = 2,\ r = s = 1$: (a) $\alpha = 0.25$; (b) $\alpha = 0.5$; (c) $\alpha = 0.75$; (d) $\alpha = 1$.

case 2: $p = 3,\ q = 3$:

According to the improved exp-function method, the solution of equation (20) in this case can be written as:

$$u(\xi) = \frac{a_0 + a_1 e^{\xi} + a_2 e^{2\xi} + a_3 e^{3\xi}}{b_0 + b_1 e^{\xi} + b_2 e^{2\xi} + b_3 e^{3\xi}}, \qquad \xi = \frac{k\, x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \qquad (24)$$

Substituting (24) into equation (20), equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations. Solving the resultant algebraic system for the unknowns $k,\ \omega,\ a_0, a_1, a_2, a_3, b_0, b_1, b_2$ and b_3, we obtain the solution sets

$$k=1,\ \omega=\frac{1}{3},\quad a_0=a_2=b_0=b_2=0,\ a_1=\frac{b_1[-24+a_3(r+s)]}{(r+s)},\quad b_3=1. \tag{25}$$

$$k=1,\ \omega=\frac{1}{8},\quad a_1=a_2=b_1=b_2=0,\ a_0=\frac{b_0[-36+a_3(r+s)]}{(r+s)},\quad b_3=1. \tag{26}$$

Thus, the solutions of the space-time fractional shallow water equation take the form

$$u_{12}(\xi)=\frac{-24b_1e^{\xi}+a_3(r+s)[b_1e^{\xi}+e^{3\xi}]}{(r+s)[b_1e^{\xi}+e^{3\xi}]},\qquad \xi=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{\alpha}}{3\Gamma(1+\alpha)}, \tag{27}$$

$$u_{13}(\xi)=\frac{-36b_0+a_3(r+s)[b_0+e^{3\xi}]}{(r+s)[b_0+e^{3\xi}]},\qquad \xi=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{\alpha}}{8\Gamma(1+\alpha)}. \tag{28}$$

When $\alpha=1$, then the results are similar to those obtained by El-Sabbagh et al [64]

3.3. The Boussinesq equation

Consider the Boussinesq equation,

$$u_{tt}-u_{xx}-u_{xxxx}-6(u_x)^2-6u\,u_{xx}=0, \tag{29}$$

which include the lowest-order effects of nonlinearity and frequency dispersion as additions to the simplest non-dispersive linear long wave theory, provide a sound and increasingly well-tested basis for the simulation of wave propagation in coastal regions. The standard Boussinesq equations for variable water depth were first derived by Peregrine (1967), who used depth-averaged velocity as a dependent variable. The space-time fractional Boussinesq equation, which is a transformed generalization of the Boussinesq equation, is defined as follows:

$$D_t^{2\alpha}u-D_x^{2\alpha}u-D_x^{4\alpha}u-6D_x^{\alpha}uD_x^{\alpha}u-6uD_x^{2\alpha}u=0,\qquad 0<\alpha\le 1, \tag{30}$$

where $u=u(x,t)$, and α is the fractional order derivative. In order to solve equation (30) by the improved exp-function method, we use the traveling wave transformation $u(x,\ t)=u(\xi)$, $\xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}$, based on this transformation, equation (30) is reduced to the following nonlinear FODE:

$$\omega^2u''-k^2u''-k^4u^{(4)}-6k^2u'^2-6k^2u\,u''=0, \tag{31}$$

where the primes denote derivatives with respect to ξ. Now we study the following cases:

case 1: $p=2,\ q=3$:

According to the improved exp-function method, the solution of equation (31) in this case can be written as:

$$u(\xi)=\frac{a_0+a_1e^{\xi}+a_2e^{2\xi}}{b_0+b_1e^{\xi}+b_2e^{2\xi}+b_3e^{3\xi}},\qquad \xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)},\quad (32)$$

Substituting (32) into equation (31), equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations. Solving the resultant algebraic system for the unknowns $k,\ \omega,\ a_0,a_1,a_2,b_0,b_1,b_2$ and b_3, we obtain the solution set

$$\omega=k\sqrt{1+k^2},\ \ a_0=a_1=b_0=0,\ \ a_2=k^2b_2,\quad b_1=\frac{b_2^2}{4},\ b_3=1.\quad (33)$$

Thus, the solutions of the space-time fractional Boussinesq equation take the form

$$u_{21}(\xi)=\frac{4k^2b_2e^{2\xi}}{b_2^2e^{\xi}+b_2e^{2\xi}+e^{3\xi}},\qquad \xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{k\sqrt{1+k^2}\,t^{\alpha}}{\Gamma(1+\alpha)}.\quad (34)$$

case 2: $p=2,\ q=4$:

According to the improved exp-function method, the solution of equation (31) in this case can be written as:

$$u(\xi)=\frac{a_0+a_1e^{\xi}+a_2e^{2\xi}}{b_0+b_1e^{\xi}+b_2e^{2\xi}+b_3e^{3\xi}+b_4e^{4\xi}},\qquad \xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)},\quad (35)$$

Substituting (35) into equation (31), equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations. Solving the resultant algebraic system for the unknowns $k,\ \omega,\ a_0,a_1,a_2,b_0,b_1,b_2,b_3$ and b_4, we obtain the solution set

$$\omega=k\sqrt{1+4k^2},\ \ a_0=a_1=b_1=b_3=0,\ \ b_0=\frac{a_2^2}{64k^4},\quad b_2=\frac{a_2}{4k^2},\ b_4=1.\quad (36)$$

Thus, the solutions of the space-time fractional Boussinesq equation take the form

$$u_{22}(\xi)=\frac{a_2e^{2\xi}}{\frac{a_2^2}{64k^4}+\frac{a_2}{4k^2}e^{2\xi}+e^{3\xi}},\qquad \xi=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{k\sqrt{1+4k^2}\,t^{\alpha}}{\Gamma(1+\alpha)}.\quad (34)$$

When $\alpha=1$, then the results are similar to those obtained by El-Sabbagh et al [64].

4. Conclusions

In this paper, the improved exp-function method is presented to find the analytical solutions of nonlinear space-time FDEs. Three examples are studied to illustrate the efficiency of the method. With the best of our knowledge, some of the obtained results are appear for the first time. The improved exp-function method can be applied to other FDEs.

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الحلول التحليلية للمعادلات التفاضلية غير الخطية الكسرية

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ملخص البحث. في هذه الورقة البحثية تم تقديم طريقة الدالة الأسية المحسنة لإيجاد حلول المعادلات التفاضلية الكسرية غير الخطية بطريقة منظمة. تم تطبيق هذه الطريقة في إيجاد حلول بعض المعادلات التفاضلية الكسرية مثل Space-time fractional Kaup–Kupershmidt equation، Space-time fractional shallow water equation، and space-time fractional Boussinesq equation. من خلال هذه الحلول فإن بعض هذه النتائج تظهر لأول مرة في هذا البحث.