Journal of Natural Sciences and Mathematics
Qassim University, Vol. 9, No. 1, pp 17-25 (January 2016/Rabi I 1437H.)

Obtaining the Efficient Solutions for Multicriterion Programming Problems with Stochastic Parameters

Adel M. Widyan

Mathematics Department, Science College, Qassim University, KSA.

Abstract. This paper deals with obtaining the set of efficient and non-dominatedfront solutions for the stochastic multicriterion programming problem(SMCPP) with random variables in both the objective functions and the right-hand side of the constraints. In this work, suggested approach uses the statistical inference in two stages, in one of them, The SMCPP is transformed into an equivalent deterministic multicriterion programming problem (DMCPP), then, in the other one, the nonnegative weighted sum approach will be applied to transform the multicriterion programming problem into a single objective programming problem.An illustrative example is presented to show the realistic implementation of the suggested approach.

Keywords: *Stochastic Multicriterion Programming, Stochastic Approach, Chance Constrained Approach.*

1. Introduction

In most of the real life problems, the Decision Maker (DM) wishes to optimize different objectives at the same time, and the values of some parameters are unknown to enable the DM to take his decision. If these unknown parameters are considered to be random variables, the resulting problem is known as stochastic multicriterion programming problem.

Stochastic multicriterion optimization has always been an interesting and challenging topic for researchers in the field of science, engineering, etc.

Stochastic programming deals with decision optimization problem where the parameters are random variable with a known probability distribution.

Stochastic programming, as an optimization method based on the probability theory, has developed in several ways (e.g. two stage problem by Dantiziq [3], chance constrained programming by Charnes and Cooper[1], especially, for multiobjective stochastic linear programming problems, Stancu-Minasian [5] considered *S*tochastic programming with multiple objective functions, which Leclereq [7] and Teghem Jr. et al. [6] proposed interactive methods.B. I. Bayoumi et al. [2] have presented an efficient approach for stochastic bi-objective programming problems with random variables.

In this paper, we consider a stochastic multicriterion programming problem (SMCPP) with stochastic parameters in both the objective functions and the right-hand side of the constraints. In this process of the solution of the stochastic problem, several mathematical and statistical tools have been used.

Therefore, the set of efficient and non-dominated solutions for SMCPP can be generated through two stagesin the first phase, The SMCPP is transformed into an equivalent deterministic multicriterion programming problem (DMCPP) using expected value criterion, then, in the other one, the nonnegative weighted sum approach, which is the one of the most familiar/popular approaches, used to convert MCPP into a single objective programming problem.

2. Multicriterion Programming Problems Formulation

Consider the following Stochastic Multicriterion Programming Problem with random variables in the objective functions and the right-hand side of the constraints as:

$$\begin{aligned} &\underset{x\in X}{\text{Min}}\ F(x) = \{f_1(x), f_2(x), \dots, f_m(x)\} \\ &\text{Subject to:} \\ &\qquad AX \geq b \\ &\qquad X \geq 0 \end{aligned} \tag{1}$$

Where, $f_i(x) = \sum_{j=1}^{n} c_{ij} x_j$, X is an n-dimensional decision variable column vector, and A is an $m \times n$ coefficient matrix, c_i, $i = 1,2,..., m$ are n-dimensional random variables row vectors. b is an m- dimensional random variables column vector, then , $AX \geq b$ can be stated as $\sum a_{ij} x_j \geq b_i$, $i = 1,2,,..., m$

belong to normal distribution with mean $E(b_i) = \mu(b_i)$ and variance $Var(b_i) = \sigma^2(b_i)$, i.e $b_i \sim N(\mu(b_i), \sigma^2(b_i))$.

Problem (1) contains stochastic parameters, therefore, definitions and solution methods for ordinary mathematical programming problems cannot be applied straight forward. Then, we consider the expected value criterionin order to solve SMCPP, along with replacing the constraints by chance-constrained conditions with satisfying a certain probability, level of significance (LOS), $\alpha_i, i = 1,2,..., m$ and the problem will be converted into an equivalent deterministic MCPP.

3. Expected Value Criterion:

By applying the expected value criterion in order to solve the SMCPP to transform it to an equivalent deterministic one, (see Hogg and Craig [4]):

$$\underset{x \in X}{\text{Min}}\ E(F(x)) = \{E(f_1(x)), E(f_2(x)), \dots, E(f_m(x))\} \quad (2)$$

The nonnegative weighted sum approach is used to transform the DMCPP into a single objective programming problem as follows:

$$\begin{aligned} \underset{x \in X}{\text{Min}}\ E\big(F(x)\big) &= \{\gamma_1 E(f_1(x)) + \gamma_2 E(f_2(x)) + \ \dots + \gamma_m E(f_m(x))\} \\ &= \left\{\gamma_1 E\left[\sum_{j=1}^{n} c_{1j} x_j\right] + \gamma_2 E\left[\sum_{j=1}^{n} c_{2j} x_j\right] + \cdots + \gamma_m E\left[\sum_{j=1}^{n} c_{mj} x_j\right]\right\} \quad (3) \\ &= \gamma_1 [x_1 E(c_{11}) + x_2 E(c_{12}) + \cdots + x_n E(c_{1n})] + \cdots \\ &\quad + \gamma_2 [x_1 E(c_{21}) + x_2 E(c_{22}) + \cdots + x_n E(c_{2n})] + \cdots \\ &\quad + \gamma_m [x_1 E(c_{m1}) + x_2 E(c_{m2}) + \cdots + x_n E(c_{mn})] \end{aligned}$$

while, $E\big(c_{ij}\big) = \mu(c_{ij}), i = 1, 2, \dots., m \ and \ \ j = 1, 2, \dots, n$

then

$$\gamma_1 \sum_{j=1}^{n} x_j \mu(c_{1j}) \quad = \gamma_1 [x_1 \mu(c_{11}) + x_2 \mu(c_{12}) + \cdots + x_n \mu(c_{1n})] \quad (4)$$

$$\gamma_2 \sum_{j=1}^{n} x_j \mu(c_{2j}) \quad = \gamma_2 [x_1 \mu(c_{21}) + x_2 \mu(c_{22}) + \cdots + x_n \mu(c_{2n})] \quad (5)$$

$$\gamma_m \sum_{j=1}^{n} x_j \mu(c_{mj}) \quad = \gamma_m [x_1 \mu(c_{m1}) + x_2 \mu(c_{m2}) + \cdots + x_n \mu(c_{mn})] \qquad (6)$$

Where $\gamma_1 + \gamma_2 + \cdots + \gamma_m = 1$, then $\gamma_m = 1 - \gamma_1 - \gamma_2 - \cdots - \gamma_{m-1}$

Therefore,

$$\gamma_m \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big) \quad = (1 - \gamma_1 - \gamma_2 - \cdots - \gamma_{m-1}) \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big)$$

$$= \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big) - \gamma_1 \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big) - \gamma_2 \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big) -$$

$$\ldots - \gamma_{m-1} \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big) \qquad (7)$$

Hence, problem (2) is transformed to an equivalent deterministic programming problem and it will be as follows:

$$\underset{x \in X}{\text{Min}}\, E\big(F(x)\big) = \sum_{j=1}^{n} x_j \mu\big(c_{mj}\big) + \sum_{i=1}^{m-1} \sum_{j=1}^{n} \gamma_i x_j [\mu\big(c_{ij}\big) - \mu\big(c_{mj}\big)] \quad (8)$$

Now, a chance constrained programming with random parameters in the right-hand side of the constraints may be stated as follows:

$$\underset{x \in X}{\text{Min}}\, E\big(F(x)\big)$$

Subject to

$$P\big[\sum_{j=1}^{n} a_{ij} x_j \geq b_i\big] \geq 1 - \alpha_i\,, \; i = 1, 2, \ldots, m \qquad (9)$$

$$x_j \geq 0 \;, \quad j = 1, 2, \ldots, n$$

Where, P means probability and α_i is a specified probability value, $b_i\,, i = 1, 2, \ldots, m$ are assumed to be normally distributed with means $E(b_i) = \mu(b_i)$ and Variances $Var(b_i) = \sigma^2(b_i)$ and independently of each other, then,

$$P\left[\frac{\sum_{j=1}^{n} a_{ij} x_j - \mu(b_i)}{\sigma(b_i)} \geq \frac{b_i - \mu(b_i)}{\sigma(b_i)}\right] \geq 1 - \alpha_i\,, \qquad i = 1, 2, \ldots, m \qquad (10)$$

Let$Z_i = \frac{b_i - \mu(b_i)}{\sigma(b_i)}$ be a random variables with standard normal distribution with mean equal zero and variance equal one, i.e.$Z_i \sim N(0, 1), i = 1, 2, \ldots, m$.

then,

$$P\left[Z_i \leq \frac{\sum_{j=1}^{n} a_{ij} x_j - \mu(b_i)}{\sigma(b_i)}\right] \geq 1 - \alpha_i\,, \qquad i = 1, 2, \ldots, m \qquad (11)$$

Therefore, $\emptyset\left[\frac{\sum_{j=1}^{n} a_{ij}x_j-\mu(b_i)}{\sigma(b_i)}\right] \geq 1-\alpha_i$, $i=1,2,\dots,m$

Where, $\phi(z)$ represents the "cumulative distribution function" of the standard normal variableZ. Let E_{α_i} represents the standard normal variable value such that $\emptyset[E_{\alpha_i}] = 1-\alpha_i$, $i=1,2,\dots,m$

Let $E_{\alpha_i} \leq \frac{\sum_{j=1}^{n} a_{ij}x_j-\mu(b_i)}{\sigma(b_i)}$, $i=1,2,\dots.,m$

Then, $\emptyset(E_{\alpha_i}) \leq \emptyset\left[\frac{\sum_{j=1}^{n} a_{ij}x_j-\mu(b_i)}{\sigma(b_i)}\right], i=1,2,\dots,m$

This inequality will be satisfied only if

$$E_{\alpha_i} \leq \left[\frac{\sum_{j=1}^{n} a_{ij}x_j-\mu(b_i)}{\sigma(b_i)}\right], i=1,2,\dots,m$$

or

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu(b_i)+E_{\alpha_i}.\sigma(b_i), i=1,2,\dots,m$$

where $\sigma(b_i)$ are the standard deviations of b_i , $i=1,2,\dots,m$.

Therefore, the equivalent deterministic problem of SMCPP can be writtenas:

$$\min_{x\in X} E(F(x)) = \sum_{j=1}^{n} x_j\mu(c_{mj}) + \sum_{i=1}^{m-1}\sum_{j=1}^{n} \gamma_i x_j[\mu(c_{ij}) - \mu(c_{mj})]$$

Subject to

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu(b_i)+E_{\alpha_i}.\sigma(b_i), i=1,2,\dots,m \qquad (12)$$

$$x_j \geq 0 \ , \quad j=1,2,\dots,n$$

Note that, if we apply the nonnegative weighted sum approach before or after using expected value criterion, the resulting problem (12) will be the same.

4. Illustrative example

To demonstrate the application of the proposed approach, consider the following stochastic multicriterion programming problem:

$$\min_{x\in X}\{c_{11}x_1+c_{12}x_2+c_{13}x_3, c_{21}x_1+\ c_{22}x_2, +c_{23}x_3c_{31}x_1+c_{32}x_{12}+c_{33}x_3\},$$

subject to

$x_1+x_2+0.5x_3 \geq b_1,$

$x_1+2x_2+5x_3 \geq b_2,$

$0.5x_1 + 2x_2 + 0.5x_3 \geq b_3,$

$x_1, x_2, x_3 \geq 0,$

Where, $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}$ are random variables with expected value as follows:

$\mu(c_{11}) = 2.0, \mu(c_{12}) = 2.5, \mu(c_{13}) = 1.5, \mu(c_{21}) = 1.5, \mu(c_{22}) = 3.0, \mu(c_{23}) = 2.5, \mu(c_{31}) = 2.5, \mu(c_{32}) = 2.0, \mu(c_{33}) = 4.0$.

Also , b_1, b_2, b_3 are normal random variables, with means $\mu(b_1) = 3.0, \mu(b_2) = 6.0, \mu(b_3) = 4.0$, and standard deviations $\sigma(b_1) = 0.5, \sigma(b_2) = 0.7, \sigma(b_3) = 0.9$, at LOS $\alpha = 0.05$, hence,$\emptyset[E_{\alpha_i}] = 1 - \alpha_i$, therefore, $E_{\alpha_i} = 1.96$.

by applying the expected value criterion method, with different weights, the subset of efficient solutions will be as shown in table (1).

Table (1). Subset of efficient and non-dominated front solutions

γ_1	γ_2	x_1	x_2	x_3	f_1	f_2	f_3	F(x)
0	0	0	3.603111	0.753778	10.13844	12.69378	10.22133	10.22133
0	1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.87467
1	0	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	9.532074
0.1	0.3	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.71223
0.2	0.2	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.57797
0.3	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.44371
0.4	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.31415
0.5	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.18459
0.6	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	10.05502
0.7	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	9.925459
0.8	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	9.795896
0.9	0.1	1.212741	2.39037	0.753778	9.532074	10.87467	10.8277	9.666333

Where; $0 \leq \gamma_i \leq 1$, $i = 1, 2$

x_1, x_2, x_3: *are the decision variavles*

f_1 : *the first objective function.*

f_2 : *the second objective function.*

f_3 : *the third objective function.*

$F(x)$: *the scalar objective function.*

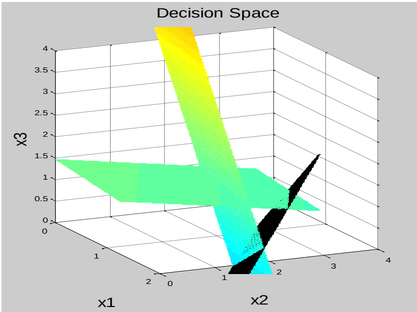

Fig. 1: Subset of efficient solution

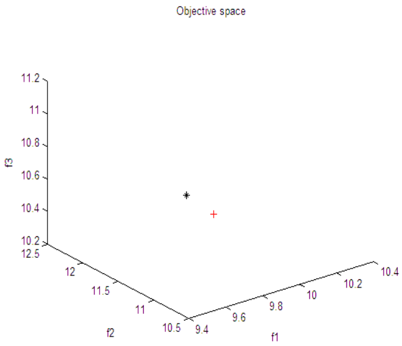

Fig. 2: Subset of non-dominated front solution

5. Discussion:

Other values of α_i, i.e. the LOS can be tested until the satisfaction of DM is fulfilled. If we use another approach for transforming the SMCPP to an equivalent DMCPP, the resulting problem will be a nonlinear MCPP.

From the Table (1), for all the values of the parameters $\gamma_i, i = 1, 2$, the set of all efficient solutions contains only two points.

it is noted that the problem is stable within some ranges of the parameters $\gamma_i, i = 1, 2$.

6. Conclusion

An approach for solving the stochastic multicriterion programming problems, with stochastic parameters in objective functions and in the right hand side of the constrains have been proposed.

An illustrative example has been provided to clarify the proposed approach and conclude that the achievement of the set of all efficient and non-dominated front solutions of SMCPP using the expected value criterion has influence in the set of efficient solutions which are obtained.

7. References

[1] A. Charnes and W.W. Cooper, Chance-constrained programming. *Management Science* 5 (1959), pp. 73-79.

[2] B. I. Bayoumi,A. A. El-Sawy, N. L. Baseley, I. K.Yousef, A.M. Widyan, Determining the Efficient Solutions for Bicriteria Programming Problems with Random Variables in Both the Objective Functions and the Constraints, *J. KSIAM Vol. 9, No. 1, 1-12, 2005*.

[3] G. B. Dantzig, Linear programming under uncertainty. *Management Science* 1 (1955), pp. 3-4.

[4] Hogg and Craig , Introduction to Mathematical Statistics. *MacMillan Publishing Co., New York* (1989), pp. 177-178.

[5] I. M. Stancu-Minasian, *Stochastic programming with multiple objective functions*. Reidel, Dordrecht (1984).

[6] J. Jr. Teghem, D. Dufrance, M. Thauvoye, P. Kunch, Strange: an interactive method for multi-objective linear programming under uncertainty, *European J. Oper. Res.* 26 (1986) 65-82.

[7] J. P. Leclereq, Stochastic Programming: An interactive multicriteria approach, *European J. Oper. Res.* 26 (1986) 31-41.

ايجاد الحلول ذات الكفاءة لمسائل البرمجة متعددة المعايير مع وجود معلمات عشوائية

عادل مفلح وديان

قسم الرياضيات، كلية العلوم، جامعة القصيم، المملكة العربية السعودية

ملخص البحث. تتناول هذه الورقة الحصول على مجموعة من الحلول ذات الكفاءة (efficient solutions) والحلول غير المهيمن عليها (efficient and non-dominated front solutions) لمشكلة البرمجة العشوائية متعددة المعايير(SMCPP) مع وجود متغيرات عشوائية في كل من دوال الهدف والطرف الأيمن من القيود. في هذا البحث، تم اقتراح منهجية تستخدم الاستدلال الإحصائي في مرحلتين، في المرحلة الأولى، يتم تحويل SMCPP إلى مشكلة البرمجة المحددة متعددة الاهداف (DMCPP)، في المرحلة الثانية، تم تطبيق منهجية مجموع الأوزان غير السالبة لتحويل مشكلة البرمجة متعددة المعايير إلى مشكلة برمجة ذات الهدف الواحد. وتم تقديم مثالا توضيحيا لعرض التنفيذ الواقعي للمنهجية المقترحة.