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The General Exact Solutions Of Linear Fractional Time-Varying Descriptor Systems In Caputo Sense

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Abstract: In this paper, we formulate the linear singular and non-singular fractional time-varying descriptor system and present the general exact solutions of both cases in Caputo sense. Furthermore, two illustrated examples are also given to show our new approach.

Keywords: *Time-Varying Descriptor System; Kronecker Product; Mittag-Leffler Matrix.*

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1. Introduction:

Matrix differential equations have been widely used in the stability, observability and controllability theories of differential equations, control theory, communication systems and many other fields of applied mathematics [1-9], and also recently in the following linear time- varying system [10-31]:

$$A(t)y^{/}(t) = B(t)y(t) + C(t)u(t) \; : \; y(t_0) = y_0 \, , \; t \geq 0 \, , \qquad (1\text{-}1)$$

where $A(t) \in M_n$ is a time-varying singular or non-singular matrix function, $B(t) \in M_m$ and $C(t) \in M_{n,m}$ are time-varying analytic matrix functions, $u(t) \in M_{m,1}$ is the output vector function and $y(t) \in M_{n,1}$ is the state function vector to be solved (where $M_{m,n}$ is denoted by the set of all $m \times n$ matrices over the real number R and when $m = n$ we write M_m instead of $M_{m,n}$). This system is usually known as a non-singular (singular) descriptor system or generalized state (semi) system or system of differential- algebraic equations and plays an important role in many applications such as in electrical networks, economics, optimization problems, analysis of control systems, engineering systems, constrained mechanics aircraft and robot dynamics, biology and large-scale systems [10-15]. The linear time- varying descriptor system as in (1-1) have been studied and discussed by many researchers [16-20]. For example, Controllability and observability of this system have been studied by Wang and Liao [17], Wang [18] and Campbell [19]; the linear of matrix differential inequalities of descriptor system was established by Inoue [20] and the stability of linear time-varying descriptor system has been discussed in [21-27]. Some special cases of the linear time-varying system as in (1-1) have been also investigated in [28-31]. For example, the stability for the special case of system (1-1) when $B(t) = B$, $C(t) = C$ and $A(t) = A$ are constant matrices has been discussed in [27-29] and also the stability analysis for the special case of system (1-1) when $B(t+T) = B(t)$, $C(t+T) = C(t)$ are periodically time-varying matrices with period T and $A(t) = A$ is a constant matrix has been studied in [24,29]. Finally, the optimal control of system as in (1-1) has been investigated in [30-31].

In addition, the topic of fractional calculus has attracted many researchers because of its several applications in various fields of applied sciences, physics and economics. For a detail survey with collections of applications in various fields, see for example [32-43] and numerous real-life problems are also modeled mathematically by systems of fractional differential equations [36,38-40,42,44-53]. Since, there are many definitions of fractional derivative of order $\alpha > 0$ and most of them used an integral or summation or limit form [e.g., 32,36,41,43,47,50-57]. One of the important and familiar definition for fractional derivative is Caputo operator which is defined by:

$$y^{\alpha}(t) = D^{\alpha} y(t) = I^{n-\alpha} D^{n} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \ , \tag{1-2}$$

where $\alpha > 0, t > 0$ and $n-1 < \alpha \le n$ ($n \in \mathbb{N}$).

Note that fractional derivative of $f(x)$ in the Caputo sense is defined for $0 < \alpha < 1$ as

$$D^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y^{/}(s)}{(t-s)^{\alpha}} ds \, . \tag{1-3}$$

Caputo's definition has the advantage of dealing property with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case most physical processes.

In the present paper, we present the general exact solutions of the singular and non-singular fractional time-varying descriptor system in Caputo sense based on the Kronecker product and vector-operator with two illustrated examples.

2. Preliminaries and Basic Concepts

In this section, we study some important basic definitions and their properties related to the Kronecker product and Mittag-Leffler function on matrices that will be useful later in our investigation of the solutions of the linear fractional time-varying descriptor systems.

Definition 2.1. Let $A = (a_{ij}) \in M_{m,n}$ and $B = (b_{kl}) \in M_{p,q}$ be two rectangular matrices. Then the Kronecker product of A and B is defined by [1-8,58-64]:

$$A \otimes B = (a_{ij} B)_{ij} \in M_{mp,nq} \ . \tag{2-1}$$

Definition 2. 2. Let $A = (a_{ij}) \in M_{m,n}$ be a rectangular matrix. Then the vector-operator of A is defined by [1-8,58-64]:

$$VecA = (a_{11} \ a_{21} \ \dots \ a_{m1} \ a_{12} \ a_{22} \ \dots \ a_{m2} \ \dots \ a_{1n} \ a_{2n} \dots \ a_{mn})^{T} \in M_{mn,1} \, , \tag{2-2}$$

Lemma. 2.1. Let A, B, C, D and X be compatible matrices in orders. Then we have [1-3, 7, 58-64].

$$\text{(i)} \ Vec(AXB) = (B^{T} \otimes A) VecX \ , \tag{2-3}$$

$$\text{(ii)} \ (A \otimes B)(C \otimes D) = AC \otimes BD \, , \tag{2-4}$$

(iii) If f is analytic function on the region containing the eigenvalues of $A \in M_m$ such that $f(A)$ exist. Then

$$f(A \otimes I_n) = f(A) \otimes I_n \quad \text{and} \quad f(I_n \otimes A) = I_n \otimes f(A). \qquad (2\text{-}5)$$

Definition 2. 3. The one-parameter Mittag-Leffler function $E_\alpha(x)$ and Mittag-Leffler matrix function $E_\alpha(At^\alpha)$ are defined for $\alpha > 0$ by [32,41,50,51,55,65] :

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)} \quad \text{and} \quad E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha + 1)}, \qquad (2\text{-}6)$$

where $A \in M_n$ is a matrix of order $n \times n$ and $\Gamma(.)$ is the Gamma function.

Lemma. 2.2. Let $A \in M_n$ be a matrix of order $n \times n$ and let $\{x_1, x_2, \cdots, x_m\}$ and $\{y_1, y_2, \cdots, y_m\}$ be the eigenvectors corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ of A and A^T, respectively. Then the spectral decomposition of $E_\alpha(A)$ and $E_\alpha(At^\alpha)$ are given, respectively, for $\alpha > 0$ by [55]:

$$E_\alpha(A) = \sum_{k=0}^{m} x_k y_k^T E_\alpha(\lambda_k) \quad \text{and} \quad E_\alpha(At^\alpha) = \sum_{k=0}^{m} x_k y_k^T E_\alpha(\lambda_k t^\alpha), \qquad (2\text{-}7)$$

The list of nice properties for Mittag-Leffler matrix $E_\alpha(A)$ can be found in [55], and the most important properties for Mittag-Leffler matrix $E_\alpha(A)$ that will be used in our study are given below [55].

Theorem. 2.1. Let $A, B \in M_m$ and I_n be an identity matrix of order $n \times n$. Then for $\alpha > 0$, we have [55]:

(i) If $A = diag(a_{11}, a_{22}, \cdots, a_{mm})$, then

$$E_\alpha(A) = diag\big(E_\alpha(a_{11}), E_\alpha(a_{22}), \cdots, E_\alpha(a_{mm})\big), \qquad (2\text{-}8)$$

(ii) $E_\alpha(A + B) = E_\alpha(A) E_\alpha(B)$ if and only if $AB = BA$, (2-9)

(iii) $E_\alpha(A \otimes I_n) = E_\alpha(A) \otimes I_n$ and $E_\alpha(I_n \otimes A) = I_n \otimes E_\alpha(A)$. (2-10)

Lemma. 2.3. Let $H(t) \in M_n$ be a given matrix function, $u(t) \in M_{n,1}$ be a given vector, and $y(t) \in M_{n,1}$ be the unknown vector to be solved. Then the unique solution of the following fractional differential system [50,51,55]:

$$y^{\alpha}(t) = H(t)y(t) + u(t) \; : \; y(0) = y_0 , \tag{2-11}$$

is given by

$$y(t) = E_{\alpha}(H(t)t^{\alpha})y_0 + \int_0^t (t-z)^{\alpha-1} E_{\alpha}\left(H(t)(t-z)^{\alpha}\right) u(z)dz . \tag{2-12}$$

3. Main Results

In this Section, we present the general exact solutions of the linear singular and non-singular fractional time-varying descriptor systems in Caputo sense based on the Kronecker product and vector-operator with two illustrated examples.

Problem 3.1. (Linear Singular Fractional Time-Varying Descriptor System)

The linear singular fractional time-varying descriptor system is formulated by

$$A(t)y^{\alpha}(t) = B(t)y(t) + C(t)u(t) \; : \; y(0) = y_0 , \; t \geq 0 , \;\; \alpha > 0 \tag{3-1}$$

where $A(t) \in M_n$ is a time-varying singular matrix function, $B(t) \in M_n$ and $C(t) \in M_{n,m}$ are time-varying analytic matrix functions, $u(t) \in M_{m,1}$ is the output vector function and $y(t) \in M_{n,1}$ is the state function vector to be solved.

For system (3-1), suppose that the constant invertible matrices M and $N \in M_n$ such that:

$$A(t) = M^{-1}\begin{bmatrix} I(t) & 0 \\ 0 & 0 \end{bmatrix}N^{-1} \;\; , \; B(t) = M^{-1}\begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}N^{-1} \; ,$$

$$C(t) = M^{-1}\begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} \text{ and } \; y(t) = N\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} . \tag{3-2}$$

This system is restricted equivalent to:

$$y_1^{\alpha}(t) = B_{11}(t)y_1(t) + B_{12}(t)y_2(t) + C_1(t)u(t)\,,$$

$$0 = B_{21}(t)y_1(t) + B_{22}(t)y_2(t) + C_2(t)u(t)\,. \qquad (3\text{-}3)$$

Note that the necessary and sufficient condition for the existence of the solution of a system (3-1) is that $B_{22}(t)$ is invertible.

General Solutions of Problem 3.1

Since $B_{22}(t)$ is invertible and from equation 2 of (3-3) we get:

$$y_2(t) = -B_{22}^{-1}(t)B_{21}(t)y_1(t) - B_{22}^{-1}C_2(t)u(t)\,. \qquad (3\text{-}4)$$

Substitute this equation in the first equation of (3-3), we get:

$$y_1^{\alpha}(t) = S_{B_{11}}(t)y_1(t) + R(t)u(t)\,, \qquad (3\text{-}5)$$

where

$$S_{B_{11}}(t) = B_{11}(t) - B_{12}(t)B_{22}^{-1}(t)B_{21}(t) \qquad (3\text{-}6)$$

is called the *Schur complement* of $B_{11}(t)$ in a matrix $\begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}$.

Now by letting

$$R(t) = -B_{12}(t)B_{22}^{-1}(t)C_2(t) + C_1(t)\,, \qquad (3\text{-}7)$$

and taking $Vec(.)$ of both sides of (3-5), and using (2-3) in Lemma 2.1, we get:

$$Vec\left(y_1^{\alpha}(t)\right) = Vec\left(S_{B_{11}}(t)y_1(t)\right) + Vec\left(R(t)u(t)\right)$$

$$= \left(I \otimes S_{B_{11}}(t)\right) Vec\left(y_1(t)\right) + \left(I \otimes R(t)\right)Vec\left(u(t)\right). \qquad (3\text{-}8)$$

Since $y_1^{\alpha}(t)$, $y_1(t)$ and $R(t)u(t)$ are vectors, then $Vec\left(y_1^{\alpha}(t)\right) = y_1^{\alpha}(t)$, $Vec\left(y_1(t)\right) = y_1(t)$

and $Vec\left(u(t)\right) = u(t)$, then the equation as in (3-8) can be represented as:

$$y_1^{\varepsilon}(t) = \left(I \otimes S_{B_{11}}(t)\right)y_1(t) + \left(I \otimes R(t)\right)u(t)\,. \qquad (3\text{-}9)$$

Now by using Lemma 2.3, then the solution of (3-9) is given by:

$$y_1(t)= E_\alpha((I\otimes S_{B_{11}}(t))\ t^\alpha)N\begin{bmatrix} y_1(0)\\ y_2(0)\end{bmatrix}+\int_0^t (t-z)^{\alpha-1}E_\alpha\Big((I\otimes S_{B_{11}}(t))(t-z)^\alpha\Big)\Big((I\otimes R(z))\ u(z)\Big)dz \quad (3\text{-}10)$$

where $S_{B_{11}}(t)$ and $R(t)$ are defined as in (3-6) and (3-7), respectively.

For appropriate the order of solutions, we take the $Vec(.)$ of both sides of (3-4) and then we get:

$$Vec\big(y_2(t)\big)=-Vec\Big(B_{22}^{-1}(t)B_{21}(t)y_1(t)\Big)-Vec\Big(B_{22}^{-1}C_2(t)u(t)\Big)$$

Which is equivalent to:

$$y_2(t)=-\Big(I\otimes B_{22}^{-1}(t)B_{21}(t)\Big)y_1(t)-\Big(I\otimes B_{22}^{-1}C_2(t)\Big)u(t)\,, \qquad (3\text{-}11)$$

where $y_1(t)$ is defined as in (3-10).

Hence, $y_1(t)$ and $y_2(t)$ as in (3-10) and (3-11), respectively, are the general exact solutions of Problem 3.1.

Problem 3.2. (Linear Non-Singular Fractional Time-Varying Descriptor System)

The linear non-singular matrix fractional time-varying descriptor system is formulated by

$$A(t)Y^\alpha(t)=B(t)Y(t)+C(t)U(t)\ :\ Y(0)=Y_0\,,\ t\geq 0\,,\ \ \alpha>0\,, \qquad (3\text{-}12)$$

where $A(t)\in M_n$ is a time-varying non-singular matrix function, $B(t)\in M_m$ and $C(t)\in M_{n,m}$ are time-varying analytic matrix functions, $U(t)\in M_m$ is the output matrix function and $Y(t)\in M_{n,m}$ is the state function matrix to be solved.

Suppose that the constant invertible matrices M and $N\in M_n$ such that:

$$A(t)=M^{-1}\begin{bmatrix} I(t) & 0\\ 0 & I(t)\end{bmatrix}N^{-1}\ ,\ B(t)=M^{-1}\begin{bmatrix} B_{11}(t) & B_{12}(t)\\ B_{21}(t) & B_{22}(t)\end{bmatrix}N^{-1}\ ,$$

$$C(t)=M^{-1}\begin{bmatrix} C_1(t)\\ C_2(t)\end{bmatrix}\ \text{and}\ \ Y(t)=N\begin{bmatrix} Y_1(t)\\ Y_2(t)\end{bmatrix}. \qquad (3\text{-}13)$$

This system is restricted equivalent to:

$$Y_1^{\alpha}(t) = B_{11}(t)Y_1(t) + B_{12}(t)Y_2(t) + C_1(t)U(t)\,,$$
$$Y_2^{\alpha}(t) = B_{21}(t)Y_1(t) + B_{22}(t)Y_2(t) + C_2(t)U(t)\,. \tag{3-14}$$

General Solutions of Problem 3.2.

By taking $Vec(.)$ of both sides of (3-14), and using (2-3) in Lemma 2.1, we get:

$$Vec\left(Y_1^{\alpha}(t)\right) = Vec\left(B_{11}(t)Y_1(t) + B_{12}(t)Y_2(t) + C_1(t)U(t)\right)$$
$$= \left(I \otimes B_{11}(t)\right)\, Vec\left(Y_1(t)\right) + \left(I \otimes B_{12}(t)\right)\, Vec\left(Y_2(t)\right) + \left(I \otimes C_1(t)\right)\, Vec\left(U(t)\right),$$
$$Vec\left(Y_2^{\alpha}(t)\right) = Vec\left(B_{21}(t)Y_1(t) + B_{22}(t)Y_2(t) + C_2(t)U(t)\right)$$
$$= \left(I \otimes B_{21}(t)\right)\, Vec\left(Y_1(t)\right) + \left(I \otimes B_{22}(t)\right)\, Vec\left(Y_2(t)\right) + \left(I \otimes C_2(t)\right)\, Vec\left(U(t)\right) \tag{3-15}$$

This system can be represented as:

$$\begin{bmatrix} Vec\left(Y_1^{\alpha}(t)\right) \\ Vec\left(Y_2^{\alpha}(t)\right) \end{bmatrix} = \begin{bmatrix} I \otimes B_{11}(t) & I \otimes B_{12}(t) \\ I \otimes B_{21}(t) & I \otimes B_{22}(t) \end{bmatrix} \begin{bmatrix} Vec\left(Y_1(t)\right) \\ Vec\left(Y_2(t)\right) \end{bmatrix} + \begin{bmatrix} \left(I \otimes C_1(t)\right) Vec\left(U(t)\right) \\ \left(I \otimes C_2(t)\right) Vec\left(U(t)\right) \end{bmatrix}. \tag{3-16}$$

Suppose that

$$T^{\alpha}(t) = \begin{bmatrix} Vec\left(Y_1^{\alpha}(t)\right) \\ Vec\left(Y_2^{\alpha}(t)\right) \end{bmatrix},\ H(t) = \begin{bmatrix} I \otimes B_{11}(t) & I \otimes B_{12}(t) \\ I \otimes B_{21}(t) & I \otimes B_{22}(t) \end{bmatrix},\ T(t) = \begin{bmatrix} Vec\left(Y_1\ (t)\right) \\ Vec\left(Y_2\ (t)\right) \end{bmatrix},$$
$$D(t) = \begin{bmatrix} \left(I \otimes C_1(t)\right) Vec\left(U(t)\right) \\ \left(I \otimes C_2(t)\right) Vec\left(U(t)\right) \end{bmatrix}.$$

Now the system as in (3-16) can be rewritten as follows:

$$T^{\alpha}(t) = H(t)T(t) + D(t) \ :\ T(0) = N^{-1}\begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix}. \tag{3-17}$$

Now by using Lemma 2.3, then the solution of (3-17) is given by:

$$T(t) = E_{\alpha}(H(t)\ t^{\alpha})\ T(0) + \int_0^t (t-z)^{\alpha-1} E_{\alpha}\left((H(t))(t-z)^{\alpha}\right)\left(D(z)\right)dz\,. \tag{3-18}$$

This leads to the following general vector solution of Problem 3.2:

$$\begin{bmatrix} Vec\left(Y_1\ (t)\right) \\ Vec\left(Y_2\ (t)\right) \end{bmatrix} = E_{\alpha}\left(\begin{bmatrix} I \otimes B_{11}(t) & I \otimes B_{12}(t) \\ I \otimes B_{21}(t) & I \otimes B_{22}(t) \end{bmatrix} t^{\alpha}\right)\ N^{-1}\begin{bmatrix} Vec\left(Y_1(0)\right) \\ Vec\left(Y_2(0)\right) \end{bmatrix}$$

$$+\int_0^t (t-z)^{\alpha-1} E_\alpha\left(\begin{bmatrix} I\otimes B_{11}(t) & I\otimes B_{12}(t) \\ I\otimes B_{21}(t) & I\otimes B_{22}(t)\end{bmatrix}(t-z)^\alpha\right)\begin{bmatrix}(I\otimes C_1(z))Vec(U(z)) \\ (I\otimes C_2(t))Vec(U(z))\end{bmatrix} dz \,. \quad (3\text{-}19)$$

The main problem in the solution of Problem 3.3 as in (3-19) is how to compute the following Mittag-Leffler matrix:

$$E_\alpha\left(\begin{bmatrix} I\otimes B_{11}(t) & I\otimes B_{12}(t) \\ I\otimes B_{21}(t) & I\otimes B_{22}(t)\end{bmatrix}\right). \quad (3\text{-}20)$$

As a special case, if $B_{11}B_{12} = B_{12}B_{22}$ and $B_{21}B_{11} = B_{22}B_{21}$, then the following matrices: $\begin{bmatrix} I\otimes B_{11}(t) & 0 \\ 0 & I\otimes B_{22}(t)\end{bmatrix}$ and $\begin{bmatrix} 0 & I\otimes B_{12}(t) \\ I\otimes B_{21}(t) & 0\end{bmatrix}$ are commutative.

Then by (2-9) of Theorem 2.3 and the same procedure in the proof of Theorem 2 in [55] , we have

$$E_\alpha\left(\begin{bmatrix} I\otimes B_{11}(t) & I\otimes B_{12}(t) \\ I\otimes B_{21}(t) & I\otimes B_{22}(t)\end{bmatrix}\right) =$$

$$\begin{bmatrix} E_\alpha(I\otimes B_{11}(t))\left\{\dfrac{E_\alpha(I\otimes B_{12}(t))+E_\alpha(-I\otimes B_{21}(t))}{2}\right\} & E_\alpha(I\otimes B_{11}(t))\left\{\dfrac{E_\alpha(I\otimes B_{12}(t))-E_\alpha(-I\otimes B_{21}(t))}{2}\right\} \\ E_\alpha(I\otimes B_{22}(t))\left\{\dfrac{E_\alpha(I\otimes B_{12}(t))-E_\alpha(-I\otimes B_{21}(t))}{2}\right\} & E_\alpha(I\otimes B_{22}(t))\left\{\dfrac{E_\alpha(I\otimes B_{12}(t))+E_\alpha(-I\otimes B_{21}(t))}{2}\right\}\end{bmatrix}$$

$$=\begin{bmatrix} I\otimes E_\alpha(B_{11}(t))\left\{\dfrac{I\otimes E_\alpha(B_{12}(t))-I\otimes E_\alpha(B_{21}(t))}{2}\right\} & I\otimes E_\alpha(B_{11}(t))\left\{\dfrac{I\otimes E_\alpha(B_{12}(t))+I\otimes E_\alpha(B_{21}(t))}{2}\right\} \\ I\otimes E_\alpha(B_{22}(t))\left\{\dfrac{I\otimes E_\alpha(B_{12}(t))+I\otimes E_\alpha(B_{21}(t))}{2}\right\} & I\otimes E_\alpha(B_{22}(t))\left\{\dfrac{I\otimes E_\alpha(B_{12}(t))-I\otimes E_\alpha(B_{21}(t))}{2}\right\}\end{bmatrix} \quad (3\text{-}21)$$

Now by substituting (3-21) in (3-19), then we get the general solution of this special case.

Example 3.1. Consider the following linear singular fractional time-varying descriptor system:

$$A(t)y^\alpha(t) = B(t)y(t) + C(t)u(t) \;:\; y(0) = y_0,\; t\ge 0,\;\; \alpha > 0, \quad (3.22)$$

where $M = N = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $A(t) = \begin{bmatrix} I_2(t) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$$B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} = \begin{bmatrix} I_2(t) & I_2 \\ I_2 & I\ (t) \end{bmatrix} = \begin{bmatrix} t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, C(t) = \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -t \\ 1 \\ t \end{bmatrix}$$

$$, u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since $S_{B_{11}}(t) = I_2(t) - I_2 I_2^{-1} I_2 = I_2(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$, $R(t) = -I_2 I_2^{-1} \begin{bmatrix} 1 \\ -t \end{bmatrix} + \begin{bmatrix} 1 \\ -t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Then the general solutions of this system by using (3-10) and (3-11), respectively, is given by:

$$y_1(t) = E_\alpha\left(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \otimes \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} t^\alpha\right)\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \int_0^t (t-z)^{\alpha-1} E_\alpha\left(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \otimes \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}(t-z)^\alpha\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} dz$$

$$= E_\alpha\left(\begin{bmatrix} t^{\alpha+2} & 0 & 0 & 0 \\ 0 & t^{\alpha+2} & 0 & 0 \\ 0 & 0 & t^{\alpha+2} & 0 \\ 0 & 0 & 0 & t^{\alpha+2} \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_\alpha\left(t^{\alpha+2}\right) & 0 & 0 & 0 \\ 0 & E_\alpha\left(t^{\alpha+2}\right) & 0 & 0 \\ 0 & 0 & E_\alpha\left(t^{\alpha+2}\right) & 0 \\ 0 & 0 & 0 & E_\alpha\left(t^{\alpha+2}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_\alpha\left(t^{\alpha+2}\right) \\ 0 \\ E_\alpha\left(t^{\alpha+2}\right) \\ 0 \end{bmatrix}.$$

Example 3.2. Consider the following linear non-singular fractional time-varying descriptor system:

$$A(t)Y^{\alpha}(t) = B(t)Y(t) \; : \; Y(0) = Y_0 \, , \; t \geq 0 \, , \;\; \alpha > 0 \, , \qquad (3\text{-}23)$$

where $M = N = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $A(t) = \begin{bmatrix} I_2(t) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix}$,

$B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} = \begin{bmatrix} I_2(t) & I_2 \\ I_2 & I\;(t) \end{bmatrix} = \begin{bmatrix} -t & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -t & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, and $Y(0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix}$.

Since $B_{11}B_{12} = B_{12}B_{22}$ and $B_{21}B_{11} = B_{22}B_{21}$, then by applying (3-19), (3-21) and Theorem 2.1, we get:

$$\begin{bmatrix} Vec(Y_1\;(t)) \\ Vec(Y_2\;(t)) \end{bmatrix} = E_{\alpha}\left(\begin{bmatrix} I_2 \otimes B_{11}(t) & I_2 \otimes I_2 \\ I_2 \otimes I_2 & I_2 \otimes B_{22}(t) \end{bmatrix} t^{\alpha} \right) \begin{bmatrix} Vec(Y_1(0)) \\ Vec(Y_2(0)) \end{bmatrix} =$$

$$\begin{bmatrix} I_2 \otimes E_{\alpha}(B_{11}(t)) \left\{ \dfrac{I_2 \otimes E_{\alpha}(I_2) - I_2 \otimes E_{\alpha}(I_2)}{2} \right\} & I_2 \otimes E_{\alpha}(B_{11}(t)) \left\{ \dfrac{I_2 \otimes E_{\alpha}(I_2)) + I_2 \otimes E_{\alpha}(I_2)}{2} \right\} \\ I_2 \otimes E_{\alpha}(B_{22}(t)) \left\{ \dfrac{I_2 \otimes E_{\alpha}(I_2) + I_2 \otimes E_{\alpha}(I_2)}{2} \right\} & I_2 \otimes E_{\alpha}(B_{22}(t)) \left\{ \dfrac{I_2 \otimes E_{\alpha}(I_2) - I_2 \otimes E_{\alpha}(I_2)}{2} \right\} \end{bmatrix} t^{\alpha}$$

$$\times \begin{bmatrix} Vec(Y_1(0)) \\ Vec(Y_2(0)) \end{bmatrix}.$$

Now,

$$\begin{aligned} Vec(Y_1(t)) &= (I_2 \otimes E_{\alpha}(B_{11}(t)))(I_2 \otimes E_{\alpha}(I_2 t^{\alpha}))\; Vec(Y_2(0)) \\ &= (I_2 \otimes E_{\alpha}(B_{11}(t)) E_{\alpha}(I_2) t^{\alpha})\; Vec(Y_2(0)) \\ &= Vec\{(E_{\alpha}(B_{11}(t)) E_{\alpha}(I_2) t^{\alpha})\; Y_2(0)\} \end{aligned}$$

That is by using (2-9), we have

$$Y_1(t)=\left(E_\alpha\left(B_{11}(t)\right)E_\alpha\left(I_2\right)t^\alpha\right)\ Y_2(0)\ =\left\{E_\alpha\left(\left(B_{11}(t)+E_\alpha\left(I_2\right)\right)t^\alpha\right)\right\}\ Y_2(0)$$

$$Y_1(t)=\begin{bmatrix} E_\alpha\left(t^\alpha-t^{\alpha+1}\right) & 0 \\ 0 & E_\alpha\left(2t^\alpha\right)\end{bmatrix}Y_2(0)\ . \tag{3-24}$$

Similarly, we get that

$$Y_2(t)=\begin{bmatrix} E_\alpha\left(t^\alpha-t^{\alpha+1}\right) & 0 \\ 0 & E_\alpha\left(2t^\alpha\right)\end{bmatrix}Y_1(0)\ . \tag{3-25}$$

Hence, the general solutions of (3-23) is given as in (2-24) and (3-25).

4. Conclusion

The general exact solutions of the linear singular and non-singular fractional time-varying descriptor systems in Caputo sense are presented by a new attractive method with two illustrated examples. How to find the sufficient conditions, stability, controllability and observability of these problems require further research.

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الحلول العامة والمضبوطة لأنظمة واصف الوقت المتغيرة الخطية من الرتب الكسرية ضمن تعريف مؤثر كابتو

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ملخص البحث. في هذه الورقة البحثية تم اعادة صياغة أنظمة واصف الوقت المتغيرة الخطية لتشمل الرتب الكسرية مع تقديم الحلول العامة والمضبوطة للوصف الجديد بشقيها المنفردة وغير المنفردة ضمن تعريف مؤثر كابتو الكسري . اضلفة الى ذلك : قمنا بتقديم بعض الامثلة التوضيحية لبيان فعالية الطريقة الجديدة والمقترحة في ايجاد هذه الحلول.