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On the Existence of Positive Solutions for the 5th Order Differential Equation for Boundary Value Problems

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Abstract. We are considering the problem of solving a nonlinear differential equation in the Banach space of real functions which are continuous on a bounded and closed interval. By means of the fixed point theory for a strict set contraction operator, this paper investigates the existence, nonexistence, and multiplicity of positive solutions for a nonlinear higher order boundary value problem.

Keywords: Positive solutions***;*** fixed-point theorem, Operator equations.

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1. INTRODUCTION

This paper investigates the existence and nonexistence of positive solutions for the following higher order boundary value problem with the help of suitable theorems in order to show the solvability nonlinear 5th order differential equation. On the other hand, the existent results of positive solutions for integer order differential equations have been studied by several researchers (see[6-9]), but, as far as we know, only a few papers consider the BVP for higher order nonlinear differential equations in the Banach space of real functions which are continuous on a bounded and closed interval, (see[1,3,5]). So, the aim of this paper is to fill this gap. In this paper, we obtain the existence and nonexistence of positive solution for the BVP in the Banach space. The results presented in this paper seem to be new and original. They generalize several results obtained up to now in the study of nonlinear differential equations of several types.

2.NOTATION, DEFINITION, AND AUXILIARY RESULTS

Theorem 2. 1 [2,4]

Assume that U is a relatively open subset of convex set K in Banach space E. Let $N:\overline{U} \to K$ be a compact map with $o \in U$. Then either

(i) N has a fixed point in $\overline{U}$;or

(ii)There is a $u \in U$ and a $\lambda \in (0,1)$ such that $u = \lambda N u$.

Definition 2.1 An operator is called completely continuous if it is continuous and maps bounded sets into precompacts.

Definition 2.2 *Let E be a real Banach space. A nonempty closed convex set* $K \subset E$ *is called cone of E if it satisfies the following conditions:*

(i) $x \in K, \sigma \geq o$ *implies* $\sigma x \in K$ *;*

(ii) $x \in K, -x \in K$ *implies* $x = o$.

3. MAIN RESULT

In this section, we will study the existence and nonexistence of positive solutions for the nonlinear boundary value problem:

$$u^{(5)}(t) = f(t,u(t)), \qquad 0 \prec t \prec 1, \qquad (3.1)$$

$$u'(1) = u''(0) = u'''(0) = u^{(4)}(0) = 0, \tag{3.2}$$

$$\alpha u(0) + \beta u'(0) = 0, \quad \text{where } \alpha, \beta \geq 0, \quad \alpha + \beta \succ 0 \tag{3.3}$$

This is equivalent for integral equation:

$$u(t) = \int_0^1 G(t,s) f(s,u(s))ds,$$

$$G = \begin{cases} \dfrac{\beta}{\alpha}\dfrac{(1-s)^3}{3!} - t\dfrac{(1-s)^3}{3!} + \dfrac{(t-s)^4}{4!}, & 0 \leq s \leq t \\ \dfrac{\beta}{\alpha}\dfrac{(1-s)^3}{3!} - t\dfrac{\beta}{\alpha}\dfrac{(1-s)^3}{3!}, & t \leq s \leq 1 \end{cases}$$

Theorem 3.1.Under conditions(3.2)and (3.3),equation(3.1)has a unique solution.

Proof. Applying the Laplace transform to equation (3.1) we get

$$s^5\bar{u}(s) - s^4 u(0) - s^3 u'(0) - s^2 u''(0) - s u'''(0) - u^4(0) = \bar{y}(s) \tag{3.4}$$

$$s^5\bar{u}(s) - s^4 A + s^3 \frac{\alpha}{\beta} A - s^2 u''(0) - s u'''(0) - u^4(0) = \bar{y}(s)$$

Where $\bar{u}(s)$ and $\bar{\text{y}}(\text{s})$ is the Laplace transform of $u(t)$ and y(t) respectively. The laplace inversion of Eq.(3.4) gives the final solution as:

$$u(t) = \frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^3}{3!} f(s,u(s))ds - \int_0^1 t\frac{(1-s)^3}{3!} f(s,u(s))ds + \int_0^t \frac{(t-s)^4}{4!} f(s,u(s))ds \tag{3.5}$$

The proof is complete.

Defining $T : X \to X$ as:

$$Tu(t)=\frac{\beta}{\alpha}\int_0^1\frac{(1-s)^3}{3!}f(s,u(s))ds-\int_0^1 t\,\frac{(1-s)^3}{3!}f(s,u(s))ds+\int_0^t\frac{(t-s)^4}{4!}\,f(s,u(s))ds \tag{3.6}$$

Where X=C[0,1] is the Banach space endowed with the supper norm. We have the following result for operator T.

Lemma 3.1

Assume that $f:[0,1]\times R\to R$ is continuous function, then T is completely continuous operator.

Proof: It is easy to see that T is continuous. For $u\in M=\{u\in X;\|u\|\le l,\ l\succ 0\}$, we

$$|Tu(t)|=\left|\frac{\beta}{\alpha}\int_0^1\frac{(1-s)^3}{3!}f(s,u(s))ds-\int_0^1 t\,\frac{(1-s)^3}{3!}f(s,u(s))ds+\int_0^t\frac{(t-s)^4}{4!}\,f(s,u(s))ds\right|$$

$$\le\frac{\beta}{\alpha}\int_0^1\frac{(1-s)^3}{3!}|f(s,u(s))|ds+\int_0^1 t\,\frac{(1-s)^3}{3!}|f(s,u(s))|ds+\int_0^t\frac{(t-s)^4}{4!}|f(s,u(s))|ds$$

$$\le\frac{\beta}{\alpha}\,\frac{L}{4!}+\ \mathrm{t}\,\frac{L}{4!}+t^5\,\frac{\mathrm{L}}{5!}\quad ,$$

where $L=\max_{0\le t\le 1,\|u\|\le 1}|f(t,u(t))|+1,$

so T(M) is bounded. Next we shall show the equicontinuity of $\overline{T(M)}$. $\forall\varepsilon\succ 0, t_1\prec t_2\in[0,1]$.

Let, $\eta\prec\left\{\frac{4!\varepsilon}{2L}\right\}$ and $\gamma\prec\left\{\frac{\varepsilon 5!}{2L}\right\}$,

we have $\mathrm{t}_2-\mathrm{t}_1\prec\eta,\qquad (\mathrm{t}_2^5+\mathrm{t}_1^5)\prec\gamma$ then,

$$|Tu(t_2)-Tu(t_1)| = \left| \begin{array}{l} -\int_0^1 \frac{(1-s)^3}{3!}(t_2-t_1) f(s,u(s))ds \\ +\int_0^{t_2} \frac{(t_2-s)^4}{4!} f(s,u(s))ds - \int_0^{t_1} \frac{(t_1-s)^4}{4!} f(s,u(s))ds \end{array} \right|$$

$$\leq L(t_2-t_1)\int_0^1 \frac{(1-s)^3}{3!}ds + L\int_0^{t_2} \frac{(t_2-s)^4}{4!}ds + L\int_0^{t_1} \frac{(t_1-s)^4}{4!}ds$$

$$\leq L\frac{(t_2-t_1)}{4!} + \frac{Lt_2^{\,5}}{5!} + \frac{\mathrm{Lt}_1^{\,5}}{5!} \leq \frac{\mathrm{L}}{4!}\eta + \frac{L\gamma}{5!}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Thus $\overline{T(M)}$ is equicontinuous. The Arzela-Ascoli theorem implies that the operator T is completely continuous.

Theorem 3.2

Assume that $f:[0,1]xR \to R$ is continuous function, and there exist constants

$$0 \prec \mathrm{c}_1 \prec (\frac{4!\alpha}{\beta},\ 4!,5!), \qquad \mathrm{c}_2 \succ o, \text{ such that } |f(t,u(t))| \leq c_1|u| + c_2$$

for all $t \in [0,1]$. Then the boundary value problem (3.1), (3.2) and (3.3) has a solution.

Proof: Following[2,4], we will apply the nonlinear alternative theorem to prove that T has one fixed point.

Let $\Omega = \{u \in X; \|u\| \prec R\}$,be open subset of X, where

$$R \succ \left(6\left\{\frac{\beta}{4!\alpha}c_1, \frac{1}{4!}c_1, \frac{c_1}{5!}, \frac{\beta}{4!\alpha}c_2, \frac{1}{4!}c_2, \frac{c_2}{5!}\right\}\right).$$

We suppose that there is a point $u \in \partial\Omega$ and $c_1 \in (0,1)$ such that $u = Tu$. So, for $u \in \partial\Omega$, we have:

$$|Tu(t)| = \left| \frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^3}{3!} f(s,u(s))ds - \int_0^1 \frac{(1-s)^3}{3!} tf(s,u(s))ds + \int_0^t \frac{(t-s)^4}{4!} f(s,u(s))ds \right|$$

$$\leq \frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^3}{3!} |f(s,u(s))|ds + \int_0^1 \frac{(1-s)^3}{3!} t|f(s,u(s))|ds + \int_0^t \frac{(t-s)^4}{4!} |f(s,u(s))|ds$$

$$\leq \frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^3}{3!} (c_1|u(s)|+c_2)ds + \int_0^1 \frac{(1-s)^3}{3!} t(c_1|u(s)|+c_2)ds + \int_0^t \frac{(t-s)^4}{4!} (c_1|u(s)|+c_2)ds$$

$$\leq \frac{\beta}{4!\alpha}(c_1|u(s)|) + \frac{1}{4!}(c_1|u(s)|) + \frac{1}{5!}(c_1|u(s)|) +$$

$$+ \frac{\beta}{4!\alpha}c_2 + \frac{1}{4!}c_2 + \frac{1}{5!}c_2 < \frac{R}{6} + \frac{R}{6} + \frac{R}{6} + \frac{R}{6} + \frac{R}{6} + \frac{R}{6} = \text{R},$$

which implies that $\|T\| \neq R = \|u\|$, that is a contraction. Then the nonlinear alternative theorem implies that T has a fixed point $u \in \overline{\Omega}$, that is , problem (3.1), (3.2) and (3.3) has a solution $u \in \overline{\Omega}$.

Finally, we give an example to illustrate the results obtained in this paper.

Example: For the boundary value problem

$$u^5(t) = \frac{2u+1}{u^2+5} \tag{3.7}$$

By using Eq. (3.1) with boundary condition (3.2) and applying theorem 3.2 with $\alpha = 1$ and $\beta = 1$. Then we have $c_1 < (\frac{4!\alpha}{\beta}, 4!, 5!)$.

We conclude that problem(3.7) has a solution.

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حول وجود حلول موجبه للمعادلة التفاضلية من الرتبة الخامسة تحت شروط حدية

د سيدة عودة

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ملخص البحث. نهتم في هذا البحث بدراسة أمكانية وجود أو عدم وجود حلول موجبه للمعادلات التفاضلية الغير خطية من الرتبة الخامسة في فضاء بناخ للدوال المتصلة على فترة محدودة ومغلقة. ويمكن إثبات ذلك لمسألة القيم الحدية ذات رتب عليا باستخدام نظرية النقطة الثابتة