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Linear Algebraic Properties of 4×4 Strongly Magic Squares

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Abstract. Magic squares have turned up throughout history, some in a mathematical context, others in philosophical or religious contexts . They have always had a great influence upon mankind's attitude. Although a definitive judgment of early history of magic squares is not available, it has been suggested that magic squares probably date back to India and pre-Islamic Persian origins. A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discuss about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given. The matrix properties of 4×4 strongly magic squares, different properties of eigenvalues and eigenvectors are discussed in detail.

Keywords: Strongly Magic Square (SMS), Eigenvalues Of Strongly Magic Square, Rank and Determinant Of Strongly Magic Square.

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1. INTRODUCTION

Magic squares developed in India and Islamic World in the first millennium C.E [1], and found its way to Europe in the later Middle Ages [2] and to sub-Saharan Africa not much after [3]. The earliest unequivocal occurrence of magic square is found in a work called Kakasaputa composed by Indian mathematician Nagarjuna in the first century AD [1].In the tenth century, the Persian mathematician Buzjani has left a manuscript in which there is a magic square, which are filled by numbers in arithmetic progression [4].Magic squares generally fall into the realm of recreational mathematics [5], however a few times in the past century and more recently, they have become the interest of more serious mathematicians. A normal magic square is a square array of consecutive numbers from $1 \ldots \mathrm{n}^2$ where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant [6]. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical properties.

2. NOTATIONS AND MATHEMATICAL PRELIMINARIES

2.1) Magic Square

A magic square of order n over a field R where R denotes the set of all real numbers is an n$^{\text{th}}$ order matrix $[a_{ij}]$ with entries in R such that

$$\sum_{j=1}^{n} a_{ij} = \rho \qquad for \quad i = 1,2,\ldots..n \quad \ldots\ldots\ldots.(1)$$

$$\sum_{j=1}^{n} a_{ji} = \rho \qquad for \quad i = 1,2,\ldots..n \quad \ldots\ldots\ldots.(2)$$

$$\sum_{i=1}^{n} a_{ii} = \rho \ , \quad \sum_{i=1}^{n} a_{i,n-i+1} = \rho \qquad \ldots\ldots\ldots.(3)$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [7]

2.2) Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

2.3) Strongly magic square (SMS): Generic Definition

A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \quad for \quad i = 1,2, \ldots .. n^2 \tag{4}$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \quad for \quad i = 1,2, \ldots .. n^2 \tag{5}$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho \ , \qquad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \tag{6}$$

$$\sum_{l=0}^{n-1} \sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \ for \ i,j = 1,2, \ldots .. n^2 \tag{7}$$

$$where\ the\ subscripts\ are\ congruent\ modulo\ n^2$$

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)} or {M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The n^{th} order sub square sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

3. PROPOSITIONS AND THEOREMS

3.1 Eigenvalues of 4 × 4 Strongly Magic Squares
Proposition 3.1.1

The eigenvalues of 4x4 SMS are $\rho, 0, \varphi, -\varphi$, where $\varphi^2 = [(a-d)^2 - (b+c)^2 + 4(\rho b + cd \pm \rho d)]$.

Proof

The general form of a 4x4 SMS is given by

$$\begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix} \quad (9)$$

where ρ is the magic constant and $a, b, c, d \in R$. [8]

The characteristic polynomial of A is given by $|A - \lambda I| = 0$

ie,

$$\begin{vmatrix} a-\lambda & b & c & d \\ \rho & c+d-\rho-\lambda & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a-\lambda & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d-\lambda \end{vmatrix} = 0 \quad (10)$$

On simplifying (10), the characteristic polynomial becomes

$$\lambda^4 - \lambda^3\rho - \lambda^2\varphi^2 + \lambda\rho\varphi^2 = 0$$

Or in factorised form; $\lambda(\lambda-\rho)(\lambda-\varphi)(\lambda+\varphi) = 0;\ where\ \varphi^2 = [(a-d)^2 - (b+c)^2 + 4(\rho b + cd \pm \rho d)]$

This completes the proof

Proposition 3.1.2

Sum of the eigenvalues of a SMS of order $n^2 \times n^2$ is the magic constant.

Proof

Let λ be the eigenvalue of a SMS.

Then $\sum \lambda_i = trace\ of\ A\ ; i = 1,2 \dots n^2$

$$\sum \lambda_i = \rho\ (Since\ trace\ of\ A\ is\ the\ magic\ constant)$$

Thus sum of the eigenvalues of a SMS is the magic constant ρ.

Proposition 3.1.3

Let $e = (e_1, e_2, \dots . e_n)^T$ where $e_i = 1 \ \forall \ i = 1,2 \dots n$,then e is the eigenvector corresponding to the eigenvalue ρ of a strongly magic square A of order n

Proof

The eigenvector X of a matrix A with eigenvalue λ is given by AX = λX.

By using the fact that the one of the eigenvalue is ρ and the row sum is also ρ; we have

$e = (e_1, e_2, \dots . e_n)^T$ as the eigenvector corresponding to eigenvalue ρ. The below given illustration clarifies the proof.

Illustration

The particular 4x4 SMS Sri Rama Chakra is given by

$$A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix}$$

For $X = (1,1,1,1)^T$ to be an eigenvector, corresponding to λ,

$$AX = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 34 \\ 34 \\ 34 \\ 34 \end{bmatrix}$$

$$= 34 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \rho X \qquad \text{Since } \rho = 34$$

Therefore $AX = \rho X$. $\therefore X = (1,1,1,1)^T$ is the eigenvector, corresponding to $\lambda = \rho$

Remark:

1. e is the eigenvector for a strongly magic square and its transpose
2. The eigenvalues except for ρ of strongly magic square of order n may be observed to be either 0 or of the form $\pm\lambda$

Corollary 3.1.4

The eigenvalues of a magic square cannot be all positive.

Proof

From Remark 2, result is obtained.

Proposition 3.1.5

Let $A = [a_{ij}]$ be an strongly magic square of order $n^2 \times n^2$ and if $B = A - \left(\frac{\rho}{n^2}\right) U$ where $U = [a_{ij}]$ such that $a_{ij} = 1 \ \forall \ i.j = 1,2 \ldots n^2$, then A and B have same eigenvalues except for ρ as it is replaced by 0 in the eigenvalue spectrum of B.

Proof

It is clear that $B = A - \left(\frac{\rho}{n^2}\right) U$ is a strongly magic square with magic constant 0. Let

$e = (e_1, e_2, \ldots . e_n)^T$ where $e_i = 1 \ \forall \ i = 1,2 \ldots n^2$ be the unit vector then $U = ee^T$

The proof will consider two different cases for the eigenvalue λ of A

Case 1: Let λ = ρ

Then $e = (e_1, e_2, \ldots . e_n)^T$ is the eigenvector of ρ,

$$Be = Ae - \left(\frac{\rho}{n^2}\right) ee^T e$$

$$= \rho e - \left(\frac{\rho}{n^2}\right) e(e^T e)$$

$$= \rho e - \left(\frac{\rho}{n^2}\right) e \ \times n^2$$

$$= \rho e - \rho e = 0$$

Therefore $e = (e_1, e_2, \ldots . e_n)^T$ being the eigenvector of strongly magic square A is now associated with eigenvalue 0 of B

Case 2: Let λ ≠ ρ be any other eigenvalue of A with another eigenvector X

Then $BX = AX - \left(\frac{\rho}{n^2}\right) ee^T X$

$BX = \lambda X - 0$ (Since X is orthogonal to e^T)

Thus $BX = \lambda X$

Therefore A and B have same eigenvalues except for ρ

3.2 Determinant and Rank of 4 × 4 Strongly Magic Squares

Proposition 3.2.1

The determinant of a 4 × 4 SMS A denoted by $|A|$ is always 0.

Proof

The general form of a 4×4 SMS is given by

$$A = \begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix}$$

$where\ \rho\ is\ the\ magic\ constant\ and\ a,b,c,d\ \in R$.[9]

As the matrix A has a null eigenvalue (From Proposition 3.1.1); its determinant is zero.

Corollary 3.2.2

The determinant of a transpose of a 4×4 SMS A denoted by A^T is always 0.

Proof

Since $|A| = 0$ (From Proposition 3.2.1) and $|A| = |A^T|$;

The determinant of the transpose of a 4×4 SMS A is always 0

Proposition 3.2.3

The rank of a 4×4 strongly magic square is always less than 4

Proof

The general form of a 4x4 SMS is given by

$$A = \begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix}$$

$$where\ \rho\ is\ the\ magic\ constant\ and\ a,b,c,d\ \in R.$$

As the matrix A has a null eigenvalue, then it is not injective as application from R^4 into itself and then rank A<dim R^4=4.

Proposition 3.2.4

The magic square AA^T has a null eigenvalue.

Proof

Let $B = AA^T$,then

$AA^T = [b_{ij}]$ $where\ i,j = 1,2,3,4\ such\ that$

$b_{11} = a^2 + b^2 + c^2 + d^2$

$b_{12} = a\rho + c(a - c + \rho) + b(c + d - \rho) + d(b + c - \rho)$

$b_{13} = -a\left(c - \frac{\rho}{2}\right) - c\left(a - \frac{\rho}{2}\right) - b\left(d - \frac{\rho}{2}\right) - d\left(b - \frac{\rho}{2}\right)$

$b_{14} = - a(a - c + \rho/2) - (c\rho)/2 - b(b + c - (3\rho)/2) - d(c + d - (3\rho)/2)$

$b_{21} = a\rho + c(a - c + \rho) + b(c + d - \rho) + d(b + c - \rho)$

$b_{22} = (a - c + \rho)^2 + (b + c - \rho)^2 + (c + d - \rho)^2 + \rho^2$

$b_{23} = - \rho(c - \rho/2) - (a - \rho/2)(a - c + \rho) - (b - \rho/2)(b + c - \rho) - (d - \rho/2)(c + d - \rho)$

$b_{24} = \rho(a - c + \rho/2) - (b + c - \rho)(c + d - (3\rho)/2) - (b + c - (3\rho)/2)(c + d - \rho) - (\rho * (a - c + \rho))/2$

$b_{31} = - a(c - \frac{\rho}{2}) - c(a - \frac{\rho}{2}) - b(d - \frac{\rho}{2}) - d(b - \frac{\rho}{2})$

$b_{32} = - \rho(c - \rho/2) - (a - \rho/2)(a - c + \rho) - (b - \rho/2)(b + c - \rho) - (d - \rho/2) * (c + d - \rho)$

$b_{33} = \left(a - \frac{\rho}{2}\right)^2 + \left(b - \frac{\rho}{2}\right)^2 + \left(c - \frac{\rho}{2}\right)^2 + \left(d - \frac{\rho}{2}\right)^2$

$b_{34} = \left(c - \frac{\rho}{2}\right)\left(a - c + \frac{\rho}{2}\right) + \frac{\rho\left(a - \frac{\rho}{2}\right)}{2} + \left(b - \frac{\rho}{2}\right)\left(c + d - \frac{3\rho}{2}\right) + (d - \rho/2)(b + c - (3\rho)/2$

$b_{41} = - a(a - c + \rho/2) - (c\rho)/2 - b(b + c - (3\rho)/2) - d(c + d - (3\rho)/2)$

$b_{42} = -\rho(a - c + \rho/2) - (b + c - \rho)(c + d - (3\rho)/2) - (b + c - (3\rho)/2)(c + d - \rho) - (\rho(a - c + \rho))/2$

$b_{43} = (c - \rho/2)(a - c + \rho/2) + (\rho(a - \rho/2))/2 + (b - \rho/2)(c + d - (3\rho)/2) + (d - \rho/2)(b + c - (3\rho)/2$

$b_{44} = \left(b + c - \frac{3\rho}{2}\right)^2 + \left(c + d - \frac{3\rho}{2}\right)^2 + \left(a - c + \frac{\rho}{2}\right)^2 + \rho^2/4,$

Also the determinant of B given by $|B| = |AA^T| = |A| \times |A^T| = 0$ (From Proposition 3.2.1 and Corollary 3.2.2)

Therefore AA^T has a null eigenvalue.

Proposition 3.2.5

The rank of the magic square AA^T is always less than 4

Proof

As the matrix AA^T has a null eigenvalue, (Proposition 3.2.4) then it is not injective as application from R^4 into itself and then rank $A<\dim R^4=4$.

4. CONCLUSION

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different mathematical courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described. Despite the fact that magic squares have been studied for a long time, they are still the subject of research projects. These include pure mathematical research, much of which is connected with the algebra and combinatorial geometry of polyhedra, polytopes…(see, for example, [9,10]). Physical application of magic squares is still a new topic that needs to be explored more. There are many interesting ideas for research in this field.

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