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Existence of multiple solutions for elliptic equation with singular cylindrical growth

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Abstract: In the present paper, an elliptic equation with singular cylindrical grouwth, is considered. By using the Nehari manifold and mountain pass theorem, the existence of at least four distinct solutions is obtained. The result depends crucially on the parameters k, λ,g and μ.

Keywords: Singular cylindrical grouwth, concave term, Nehari manifold, mountain pass theorem

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Introduction

In this paper, we consider the multiplicity results of nontrivial nonnegative solutions of the following problem (1.1)

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|y|^2} = |u|^{2^*-2}u + \lambda g(y)|u|^{q-2}u \ in\ \Omega, y \neq 0 \\ u = 0, \qquad on\ \partial\Omega \end{cases}$$

With $\Omega \subset \mathbb{R}^k \times \mathbb{R}^{N-k}$ where each point x in $\mathbb{R}^N$ is written as a pair (y,z)∈ $\mathbb{R}^k \times \mathbb{R}^{N-k}$ where k and N are integers such that N≥3 and k belongs to $\{2,\ldots,N\}, 2^* = 2N/(N-2)$ is the Sobolev critical exponent, 1<q<2, -∞< $\mu < \overline{\mu_k} = \frac{(k-2)^2}{4}$, λ is a real parameter and g is continuous function in $\overline{\Omega}$. In recent years, many auteurs have paid much attention to the following singular elliptic problem, i.e., the case k=N,g≡1 in (1.1), (1.2):

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{p-2}u + \lambda u \ in\ \Omega, x \neq 0 \\ u = 0, \qquad on\ \partial\Omega \end{cases}$$

Where Ω is a smooth bounded domain in

$$\mathbb{R}^N \text{ (N>2)}, 0\in \Omega, \lambda > 0, 0 \leq \mu < \overline{\mu_N} = \frac{(N-2)^2}{4}$$

And $2^* = 2N/(N-2)$ is the critical Sobolev exponent, see [5,6,8] and references therein. The quasilinear form of (1.2) is discussed in [11].Some results are already available for (1.1). Wang and Zhou [17] proved that there exist at least two solutions for (1.1) with $0 \leq \mu < \overline{\mu_N} = \frac{(N-2)^2}{4}$. Bouchekif and Matallah [2] showed the existence of two solutions of (1.1) under certain conditions on a weighted function h, when $0 \leq \mu < \overline{\mu_N}$, $0 < \lambda < \overline{\lambda}$ with $\overline{\lambda}$ a positive constant.

Concerning existence results in the case k<N, we cite [9,10,14] and the references therein. Musina [14] considered (1.1) with $\lambda = 0$, also (1.1). She established the existence of a ground state solution when 2<k≤N and 0< $\mu < \overline{\mu_k}$ for (1.1) with $\lambda = 0$. She also showed that (1.1) with $\lambda = 0$ does not admit ground state solutions. Badiale et al. [1] studied (1.1) with $\lambda = 0$. They proved the existence of at least a nonzero nonnegative weak solution u, satisfying u(y,z)=u(|y|,z) when 2≤k<N and $\mu < 0$. Bouchekif and El Mokhtar [3] proved that (1.1) admits two distinct solutions when 2<k≤N, b=N-p(N-2)/2 with 2<p≤ 2^*, $< \mu < \overline{\mu_k}$ and $0 <$

$\lambda < \bar{\lambda}$ with $\bar{\lambda}$ a positive constant. Terracini [16] proved that there is no positive solutions of (1.1) with

$\lambda = 0$ when $\mu < 0$. The regular problem corresponding to has been considered on a regular bounded domain Ω by Tarantello [15]. She proved that, with a nonhomogeneous term g$\in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notations.

We denote by $D_0^{1,2} = D_0^{1,2}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$ and $H_\mu = H_\mu \ (\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$, the closure of $C_0^\infty(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$ with respect to the norms

$$\|u\|_\mu = \left(\int_\Omega (|\nabla u|^2 - \mu|y|^{-2}|u|^2) dx\right)^{1/2} \, and \, \|u\| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}$$

respectively, with $\mu < \overline{\mu_k} \, for \, k \neq 2$.

From the Hardy-Sobolev-Maz'ya inequality, it is easy to see that the norm$\|u\|_\mu$ is equivalent to $\|u\|$. More explicitly, we have

$$(1 - (\sqrt{\overline{\mu_k}})^{-2} \mu^+)^{1/2} \|u\| \le \|u\|_\mu \le (1 - (\sqrt{\overline{\mu_k}})^{-2} \mu^-)^{\frac{1}{2}} \|u\|,$$

With $\mu^+ = \max(\mu, 0) \, and \, \mu^- = min(\mu, 0) \, for \, all \, u \in H_\mu$.

We list here a few integral inequalities.

The starting point for studying (1.1), is the Hardy inequality with cylindrical weights [14]. It states that

$$\overline{\mu_k} \int_\Omega |y|^{-2} v^2 dx \le \int_\Omega |\nabla v|^2 \, dx, for \, all \, v \in H_\mu \ ,$$

Since our approach is variational, we define the functional I on H_μ by

I(u):=(1/2) $\|u\|_\mu{}^2 - \left(\frac{1}{2^*}\right) \int_\Omega |u|^{2^*} dx - \left(\frac{\lambda}{q}\right) \int_\Omega g|u|^q dx,$

A point u$\in H_\mu$ is a weak solution of the equation (1.1) if it satisfies

$$\langle I'(u), \varphi \rangle := \int_\Omega \left((\nabla u \nabla \varphi) - \mu|y|^{-2}(u\varphi) \right) dx - \int_\Omega |u|^{2^*-2}(u\varphi) \, dx$$

-$\lambda \int_{\Omega} g|u|^{q-2}(u\varphi)dx, for\ all\ \varphi \in H_{\mu}$.

$\langle .,. \rangle$ here denotes the product in the duality $\langle H'_{\mu}\ , H_{\mu}\ \rangle$ $(H'_{\mu}\ dual\ of\ H_{\mu}\ )$

Let

$$S_{\mu}\ \coloneqq \inf_{u \in H_{\mu}\ \setminus \{0\}} \frac{\|u\|_{\mu}{}^{2}}{(\int_{\Omega}\ |u|^{p}\ dx)^{2/p}}$$

From [12], S_{μ} is achieved. Now we consider the following assumption:

(G) g is a continuous function defined in $\overline{\Omega}$ and there exist g₀ and ρ_0 positive such that g(x)≥g₀ for all x∈ $B(0, \rho_0\ )$...(

In our work, we research the critical points as the minimizers of the energy functional associated to the problem (1.1) on the constraint defined by the Nehari manifold, which are solutions of our system.

Let ρ_0 be positive number such that

$$\lambda_0\ \coloneqq (S_{\mu}\ )^{2(2-q)/2^*(2^*-2)}\ \frac{(2^*-2)}{(2^*-q)}((\frac{(2^*-2)}{(2^*-q)})^{\left(\frac{2-q}{2^*-2}\right)}\ \frac{1}{\|g\|_{H_{\mu}\ ^{-1}}}.$$

Now we can state our main results.

Theorem1: Assume that, -∞< $\mu < \overline{\mu_k}$ and λ verifying 0< $\lambda < \lambda_0$, then the system (1.1) has at least one positive solution.

Therem2: In addition to the assumptions of the Theorem1, there exists $\lambda_1\ = \frac{q}{2}\lambda_0$ such that if λ

Satisfying 0< $\lambda < \lambda_1$, then (1.1) has at least two positive solutions.

Theorem3: In addition to the assumptions of the Theorem2, assuming N≥6, there exists a positive real λ_2 such that, if λ satisfy 0< $\lambda <$ min $(\lambda_1\ , \lambda_2\ )$, then (1.1) has at least two positive solutions and at least one pair of sign-changing solutions.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last Section, we prove the Theorem3.

2. Preliminaries

Definition1: Let c∈ℝ, E a Banach space and I∈C¹(E,ℝ)

i) $(u_n\ )_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

I$(u_n) = c + o_n(1)$ and $I'(u_n) = o_n(1)$

Where $o_n(1)$ tends to 0 as n goes at infinity.

ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

Lemma1: Let X Banach space, and J∈C^1(X,R) verifying the Palais -Smale condition. Suppose that J(0)=0 and that:

i) there exist R>0, r>0 such that if ‖u‖=R, then J(u)≥r

ii) there exist (u_0)∈X such that ‖u_0‖>R and J(u_0)≤0.

Let

c:= $\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} (J(\gamma(t)))$

where

$$\Gamma = \{\gamma \in C([0,1],X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = u^0\}$$

then c is critical value of J such that c≥r.

Nehari manifold

It is well known that I is of class C^1 in H_μ and the solutions of (1.1) are the critical points of I which is not bounded below on H_μ. Consider the following Nehari manifold

N={u∈ $H_\mu \setminus \{0\}$: $\langle I'(u), u\rangle = 0$}

Thus, u∈N if and only if

$$\|u\|_\mu^{\,2} - \int_\Omega |u|^{2^*} dx - \lambda \int_\Omega g|u|^q dx = 0. (1)$$

Note that N contains every nontrivial solution of the problem (1.1). Moreover, we have the following results.

Lemma2: I is coercive and bounded from below on N.

Proof: If u∈N, then by (1) and the Hölder inequality, we deduce that

I(u):=(($2^* - 2)/ 2^*$ 2) $\|u\|_\mu^{\,2} - \lambda((2^* - q)/ 2^* q) \int_\Omega g|u|^q dx,$

≥(($2^* - 2)/ 2^*$ 2) $\|u\|_\mu^{\,2} - \lambda((2^* - q)/ 2^* q))\, \|u\|_\mu^{\,q}\, \|g\|_{H_\mu^{-1}}$. (2)

Thus, I is coercive and bounded from below on N.

Define

$$\varphi(u) = \langle I'(u), u\rangle$$

Then, for u∈N

$$\langle\varphi'(u), u\rangle = 2\,\|u\|_{\mu}^{2} - 2^{*}\int_{\Omega} |u|^{2^{*}}dx - \lambda\int_{\Omega} g|u|^{q}dx,$$

$$= (2\text{-q})\,\|u\|_{\mu}^{2} - (2^{*}-1)\int_{\Omega} |u|^{2^{*}}dx$$

$$\lambda(2^{*}-q)\int_{\Omega} g|u|^{q}dx - (2^{*}-2)\,\|u\|_{\mu}^{2}. (3)$$

Now, we split N in three parts:

$$N^{+}=\{u\in N:\ \langle\varphi'(u), u\rangle > 0\}$$

$$N^{0}=\{u\in N:\ \langle\varphi'(u), u\rangle = 0\}$$

$$N^{-}=\{u\in N:\ \langle\varphi'(u), u\rangle < 0\}$$

We have the following results.

Lemma3: Suppose that u_0 is a local minimizer for I on N. Then, if $u_0 \notin N^0$, u_0 is a critical point of I.

Proof: If u_0 is a local minimizer for I on N, then u_0 is a solution of the optimization problem

$$\min_{\{u/\varphi(u)=0\}} I(u)$$

Hence, there exists a Lagrange multipliers θ∈ℝ such that

$I'(u_0) = \theta\varphi'(u_0)$ *in* H' .

Thus,

$$\langle I'(u_0), u_0\rangle = \theta\langle\varphi'(u_0), u_0\rangle$$

But $\langle\varphi'(u_0), u_0\rangle \neq 0$, since $u_0 \notin N^0$. Hence θ=0. This completes the proof.

Lemma4: There exists a positive number λ_0 such that for all λ, verifying

$0 < \lambda < \lambda_0$

we have $N^0 \neq \emptyset$.

Proof: Let us reason by contradiction.

Suppose $N^0 \neq \emptyset$ such that $0 < \lambda < \lambda_0$. Then, by (3) and for $u \in N^0$, we have

$$(2-q)\,\|u\|_{\mu}^{2} - (2^{*}-1)\int_{\Omega} |u|^{2^{*}} dx = 0$$

$$\lambda(2^{*}-q)\int_{\Omega} g|u|^{q} dx - (2^{*}-2)\,\|u\|_{\mu}^{2}.$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\|_{\mu} \;\geq \left(S_{\mu}\right)^{\frac{2}{2^{*}(2^{*}-2)}} \left[\frac{2-q}{2^{*}-1}\right]^{\frac{1}{(2^{*}-2)}} \quad (4)$$

And

$$\|u\|_{\mu} \;\leq \left[\frac{2^{*}-q}{2^{*}-2}\right]^{\frac{1}{(2-q)}} [\lambda]^{\frac{1}{(2-q)}} \; (5).$$

From (4) and (5), we obtain $\lambda \geq \lambda_0$, which contradicts an hypothesis.

Thus $N = N^{+} \cup N^{-}$. Define

$$c := \inf_{u \in N} I(u),\; c^{+} := \inf_{u \in N^{+}} I(u),\; c^{-} := \inf_{u \in N^{-}} I(u),$$

For the sequel, we need the following Lemma.

Lemma5:

i) For all such that $0 < \lambda < \lambda_0$, one has $c \leq c^{+} < 0$

ii) For all such that $0 < \lambda < \frac{q}{2}\lambda_0$, one has

$$c^{-} > C_0 \;= C_0\left(\lambda, S_{\mu}, \|h^{+}\|_{\infty}, q\right).$$

Proof: (i) Let $u \in N^{+}$. By (3), we have

$$\left[\frac{2-q}{2^{*}-1}\right] \|u\|_{\mu}^{2} > \int_{\Omega} |u|^{2^{*}} dx$$

And so

$$I(u) = [(q-2)/2q]\,\|u\|_{\mu}^{2} + [(2^{*}-q)/2^{*}q]\int_{\Omega} |u|^{2^{*}} dx$$

$$< -(2-q)\left[\frac{2^{*}(2^{*}-1)-2(2^{*}-q)}{2q2^{*}(2^{*}-1)}\right]\|u\|_{\mu}^{2} < 0.$$

We conclude that $c \leq c^{+} < 0$.

ii) Let $u \in N^{-}$. By (3), we get

$$\left[\frac{2-q}{2^{*}-1}\right] \|u\|_{\mu}^{2} < \int_{\Omega} |u|^{2^{*}} dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_{\Omega} |u|^{2^*} dx \leq (S_\mu)^{\frac{-2^*}{2}} \|u\|_\mu^{2^*}.$$

This implies

$$\|u\|_\mu > (S_\mu)^{\frac{2^*}{2(2^*-2)}} \left[\frac{2-q}{2^*-1}\right]^{\frac{1}{(2^*-2)}}, for\ all\ u \in N^-. \ (6)$$

By (2), we get

$$I(u) \geq \|u\|_\mu{}^q \left[\frac{2^*-2}{2^*2}\right]\left[\frac{2-q}{2^*-1}\right]^{\frac{(2-q)}{(2^*-2)}} (S_\mu)^{\frac{2^*(2-q)}{2(2^*-2)}} +$$
$$-\lambda \|u\|_\mu{}^q \left[\left(\frac{2^*-q}{2^*2}\right)\|g\|_{H_\mu^{-1}}\right].$$

Thus, for all λ such that

$$0< \lambda < \lambda_1 = \left[\frac{2^*-2}{2^*2}\right]\left[\frac{2-q}{2^*-1}\right]^{\frac{(2-q)}{(2^*-2)}} (S_\mu)^{\frac{2^*(2-q)}{2(2^*-2)}} \left(\frac{2^*q}{2^*-q}\right)(1/\|g\|_{H_\mu^{-1}}) = \frac{q}{2}\lambda_0$$

we have $I(u) \geq C_0$.

Proposition1: (see [4])

i) For all λ such that $0< \lambda < \lambda_0$, there exists a $(PS)_{c^+}$ sequence in N^+

ii) For all such that $0< \lambda < \frac{q}{2}\lambda_0$, there exists a $(PS)_{c-}$ sequence in N^- .

We write

$$t_M := t_{max} = \left[\frac{(2-q)\|u\|_\mu{}^2}{(2^*-q)\int_\Omega |u|^{2^*} dx}\right]^{\frac{1}{(2^*-2)}} > 0.$$

Lemma6: Let real parameters such that $0< \lambda < \lambda_0$. For each $\in H_\mu$, there exist unique

$$t^+ \text{ and } t^- \ such\ that\ 0 < t^+ < t_m < t^-, (t^+u) \in N^+,\ (t^-u) \in N^-$$
$$I(t^+u) = \inf_{0<t<t_m} I(tu) \text{ and } I(t^-u) = \inf_{t\geq 0} I(tu).$$

Proof: With minor modifications, we refer to [4].

3. Proofs

Proof of Theorem1

Now, taking as a starting point the work of Tarantello [13], we establish the existence of a local minimum for I on N^+ .

Proposition2: For all such that $0< \lambda < \lambda_0$, the functional I has a minimizer $u_0^+ \in N^+$ and it satisfies

i) $I(u_0^+) = c = c^+$

ii) (u_0^+) is a nontrivial solution of (1.1).

Prof: If $0< \lambda < \lambda_0$, then by Proposition1 (i) there exists a $(u_n)_n$ - $(PS)_{c^+}$ sequence in N^+, thus it bounded by **Lemma2**. Then, there exists $u_0^+ \in H$ and we can extract a subsequence

which will denoted by $(u_n)_n$ such that

$$u_n \to u_0^+ \text{ weakly in } H$$

$$u_n \to u_0^+ \text{ weakly in } L^{2^*}(\Omega)$$

$$u_n \to u_0^+ \text{ strongly in } L^q(\Omega)$$

$$u_n \to u_0^+ \text{ a.e in } \Omega. \quad (7)$$

Thus, by (7), u_0^+ is a weak nontrivial solution of (1.1). Now, we show that u_n converges to u_0^+ strongly in H. Suppose otherwise. By the lower semi-continuity of the norm, then either

$$\|u_0^+\|_\mu < \liminf_{n\to\infty} \|u_n\|_\mu$$

and we obtain

$$c\ I(u_0^+) = [(2^*\text{-}2)/2^*2]\ \|u_0^+\|_\mu^{\ 2} - \lambda(2^* - q)/2^* q \int_\Omega g|u_0^+|^q dx.$$

$$< \liminf_{n\to\infty} I(u_n) = c$$

We get a contradiction. Therefore, u_n converge to u_0^+ strongly in H. Moreover, we have $u_0^+ \in N^+$. If not, then by **Lemma6**, there are two numbers t_0^+ and t_0^-, uniquely defined so that $(t_0^+ u_0^+) \in N^-$ and $(t^- u_0^+) \in N^+$. In particular, we have $t_0^- < t_0^+ = 1$. Since

$$\frac{d}{dt} I(t\ u_0^+)(t = t_0^+) = 0 \text{ and } \frac{d^2}{dt^2} I(t\ u_0^+)(t = t_0^+) > 0,$$

there exists $t_0^- < t^- \leq t_0^+$ such that $I(t_0^- u_0^+) < I(t^+ u_0^+)$. By **Lemma6**, we get

$$I(t_0^- u_0^+) < I(t^-\ u_0^+) < I(t^+ u_0^+) = I(u_0^+),$$

which contradicts the fact that $I(u_0^+)=c^+$. Since $I(u_0^+)=I(|u_0^+|)$ and $|u_0^+|\in N^+$, then by **Lemma3**, we may assume that u_0^+ is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that $u_0^+>0$, see for example [7].

Proof of Theorem2

Next, we establish the existence of a local minimum for I on N^-. For this, we require the following Lemma.

Lemma7: For all λ such that $0<\lambda<\frac{q}{2}\lambda_0$, the functional I has a minimizer u_0^- in N^- and it satisfies:

i) $I(u_0^-)=c^->0$

ii) u_0^- is a nontrivial solution of (1.1) in H.

Proof: If $0<\lambda<\frac{q}{2}\lambda_0$, then by Proposition1 (ii) there exists a $(u_n)_n$ - $(PS)_{c-}$ sequence in N^-, thus it bounded by **Lemma2**. Then, there exists $u_0^-\in H$ and we can extract a subsequence which will denoted by (u_n) such that

$$u_n \to u_0^- \text{ weakly in } H$$
$$u_n \to u_0^- \text{ weakly in } L^{2^*}(\Omega)$$
$$u_n \to u_0^- \text{ strongly in } L^q(\Omega)$$
$$u_n \to u_0^- \text{ a.e in } \Omega.$$

This implies

$$\int_\Omega |u_n|^{2^*}dx \to \int_\Omega |u_0^-|^{2^*}dx,\ as\ n\ goes\ to\ \infty$$

Moreover, by (3) we obtain

$$\int_\Omega |u_n|^{2^*}dx > [(2^*\text{-}q)/(2^*-1)]\ \|u\|_\mu^{\ 2}\ (8).$$

By (4) and (8) there exists a positive number

$$C_1 := \left[\frac{2-q}{2^*-1}\right]^{\frac{(2^*-1)}{(2^*-2)}} (S_\mu)^{\frac{2}{2^*(2^*-2)}}$$

Such that

$$\int_\Omega |u_n|^{2^*} dx > C_1 \ (9).$$

This implies that

$$\int_\Omega |u_0^-|^{2^*} dx > C_1$$

Now, we prove that (u_n) converges to u_0^- strongly in H. Suppose otherwise. Then, either

$$\|u_0^-\|_\mu < \liminf_{n\to\infty} \|u_n\|_\mu$$

By **Lemma6** there is a unique t_0^- such that $(t_0^- u_0^-) \in N^-$. Since

$$u_n \in N^-, \mathrm{I}(u_n) \geq I(tu_n), for\ all\ t \geq 0,$$

We have

$$\mathrm{I}(t_0^- u_0^-) < \lim_{n\to\infty} \mathrm{I}(t_0^- u_n) \leq \mathrm{I}(u_n) = c^-$$

and this is a contradiction. Hence, (u_n) converges to u_0^- strongly in H.

Thus $\mathrm{I}(u_n)$ *converges to* $I(u_0^-) = c^-$ as n tends to $+\infty$.

Since $\mathrm{I}(u_0^-) = \mathrm{I}(|u_0^-|)$ and $u_0^- \in N^-$, then by (9) and **Lemma6,** we may assume that u_0^- is a nontrivial nonnegative solution of (1.1). By the maximum principle, we conclude that $u_0^- > 0$.

Now, we complete the proof of **Theorem2**. By **Propositions2** and **Lemma7**, we obtain that (1.1) has two positive solutions $u_0^+ \in N^+$ and $u_0^- \in N^-$. Since $N^+ \cap N^- = \emptyset$, this implies that u_0^+ and u_0^- are distinct.

Proof of Theorem3

In this section, we consider the following Nehari submanifold of N.

$$N_r = \{u \in H_\mu \setminus \{0\}: \langle I'(u), u\rangle = 0\ and\ \|u\|_\mu \geq r > 0\}.$$

Thus, $u \in N_r$ if and only if

$$2\,\|u\|_\mu^2 - \int_\Omega |u|^{2^*} dx - \lambda \int_\Omega g|u|^q dx = 0\ and\ \|u\|_\mu \geq r > 0.$$

Firsly, we need the following Lemmas

Lemma8: Under the hypothesis of **Theorem3**, there exist $r_0, \lambda_2 > 0$ such that N_r is nonempty for any $0 < \lambda < \lambda_2$ and $0 < r < r_0$.

Proof: Fix $u_0 \in H \setminus \{0\}$ and let

$$g(t) = \langle I'(tu_0), tu_0\rangle = t^2 \|u_0\|_\mu^2 - t^{2^*} \int_\Omega |u_0|^{2^*} dx - \lambda t^q \int_\Omega g|u_0|^q dx.$$

Clearly g(0)=0 and g(t)→-∞ as t→+∞. Moreover, we have

$$g(1)= \|u_0\|_\mu^{\,2} - \int_\Omega |u_0|^{2^*}dx - \lambda\int_\Omega g|u_0|^q dx.$$

$$\geq \left[\|u_0\|_\mu^{\,2} - (S_\mu)^{\frac{-2^*}{2}}\|u_0\|_\mu^{\,2^*}\right] - \lambda\|u_0\|_\mu^{\,q}\|g\|_{H_\mu^{-1}}$$

If $\|u\|_\mu \geq r > 0\, for\ 0 < r < r_0 = (S_\mu)^{\frac{2^*}{2(2^*-2)}}$, then there exist

$$\lambda_2 := r^{2-q}(1 - r^{2^*-2}(S_\mu)^{\frac{-2^*}{2}})(\frac{1}{\|g\|_{H_\mu^{-1}}})$$

and $t_0>0$ such that $g(t_0)=0$. Thus, $(t_0 u_0) \in N_r$ and N_r is nonempty for any $0< \lambda < \lambda_2$.

Lemma9: There exist ρ, λ_2 positive reals such that $\langle\varphi'(u), u\rangle < -\rho < 0$ for u$\in N_r$ and any λ verifying

$$0< \lambda < \min(\lambda_2, \lambda_3)$$

Let u$\in N_r$, then by (1), (3) and the Holder inequality, allows us to write

$$\langle\varphi'(u), u\rangle = \lambda(2^* - q)\int_\Omega g|u|^q dx - (2^* - 2)\|u\|_\mu^{\,2}$$

$$\leq \lambda(2^* - q)\|u\|_\mu^{\,q}\|g\|_{H_\mu^{-1}} - (2^* - 2)\|u\|_\mu^{\,2}$$

$$\leq \|u\|_\mu^{\,q}[\lambda(2^* - q)\|g\|_{H_\mu^{-1}} - (2^* - 2)r^{2-q}],$$

Thus if

$$0< \lambda < \lambda_4 = [(2^* - 2)r^{2-q}/(2^* - q)\|g\|_{H_\mu^{-1}}]$$

and choosing $\lambda_3 := \min(\lambda_2, \lambda_4)$ with λ_2 defined in **Lemma9**, then we obtain that

$$\langle\varphi'(u), u\rangle < 0, for\ any\ u \in N_r\ . (10)$$

Lemma10: Suppose N≥6. Then, there exist α and η positive constants such that

i) we have I(u) ≥ η>0 for $\|u\|_\mu = \acute{\varepsilon}$.

ii) there exists w$\in N_r$ when $\|u\|_\mu > \acute{\varepsilon}$, with $\|u\|_\mu = \acute{\varepsilon}$ such that I(w)≤0.

Proof: We can suppose that the minima of J are realized by u_0^+ and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have:

i) By (3), (10) and the fact that

$$\int_{\Omega} |u|^{2^*}dx > [(2-q)/(2^*-1)\,]\|u\|_{\mu}{}^{2}$$

We get

$$\text{I(u)} \geq [((q-1)/2q) + ((2^*-1)/2^*q)((2-q)/(2^*-1))\,]\|u\|_{\mu}{}^{2}$$

By the fact that 1<q<2 and N≥6, we obtain that

$$\text{I(u)} \geq \eta > 0 \text{ when } \acute{\varepsilon} = \|u\|_{\mu} \quad small.$$

ii) Let t>0, then we have for all $\Psi \in N_r$.

$$\text{I }(t\Psi) = (t^2/2)\,\|\Psi\|_{\mu}{}^{2} - (t^{2^*}/2^*)\int_{\Omega} |\Psi|^{2^*}dx - \lambda(t^q/q)\int_{\Omega} g|\Psi|^q dx.$$

Letting w= tΨ for t large enough, we obtain I(w)≤0.For t large enough we can ensure

$\|w\|_{\mu} \; > \acute{\varepsilon}$. Let and c defined by

$$:=\{\gamma:[0,1] \to N_r : \gamma(0) = u_0{}^{-} \; and \; \gamma(1) = u_0{}^{+}\}$$

And

$$\text{c:=} \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} (I\left(\gamma(t)\right))$$

If $0 < \lambda < \min(\lambda_1, \lambda_2)$ then, by the **Lemma2** and **Proposition1** (ii), the functional I verifying the Palais -Smale condition in N_r . Moreover, from the Lemmas 3, 9 and 10, there exists u_c such that $I(u_c) = c$ and $u_c \in N_r$.

Thus $u_c \neq u_0{}^{-}$ and $u_c \neq u_0{}^{+}$ is the third solution of our system such that . Since (1.1) is odd with respect u, we obtain that $-u_c$ is also a solution of (1.1).

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