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.A Fixed Point Results Of -distance In Complete Metric Spaces

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Abstract. Using the concept of W-distance, a result on the existence of fixed points for multivalued maps is proved. Consequently, provide the previous works done by others and then compare them with the presented one explicitly, of a new condition that is deduced from the properties of W-distance.

Keywords: W-distance, Banach contraction, Kannan contraction, a lower semi-continuous, Non commutative W-contraction, generalized non commutative W-contraction and Non commutative k_w-map.

Introduction

Throughout this paper, unless otherwise specified, X is a metric space with metric d. Let 2^X, $CL(X)$ and $CB(X)$ denote the collection of nonempty subsets of X, nonempty closed subsets of X, and nonempty closed bounded subsets of X, respectively. By using the properties of w-distance and the generalized w-contraction map we prove fixed point and common fixed point results for multivalued maps in the setting of metric spaces.

Preliminaries

Consider a single valued map $f : X \to X$ and a multivalued map $T : X \to 2^X$

(i) A point $x \in X$ is called a fixed point of f if $f(x) = x$ and a fixed point of T if $x \in T(x)$.

(ii) f is called Banach contraction if for a fixed constant $k \in [0,1)$ and for each $x, y \in X$.

$$d(f(x), f(y)) \le kd(x, y)$$

(iii) f is called Kannan contraction if for a fixed constant $h \in [0, \frac{1}{2})$ and for each $x, y \in X$.

$$d(f(x), f(y)) \le h[d(x, f(x)) + d(y, f(y))]$$

Definition 1.1

A map $\psi : X \to \mathbb{R}$ is called a lower semi-continuous if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\psi(x) \le \liminf_{n\to\infty} \psi(x_n)$

Recently, Kada et al. [1] introduce a concept of w-distance as follows

Definition 1.2

A function $w : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following:

(i) $w(x,z) \le w(x, y) + w(y,z)$ for all $x, y, z \in X$.

(ii) $w(x,.) : X \to [0,\infty)$ is a lower semicontinuous map, i.e, if a sequence $\{y_n\}$ in X with $y_n \to y \in X$, then $w(x, y) \le \liminf_{n\to\infty} w(x, y_n)$.

(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$

imply $d(x,y) \le \varepsilon$.

The metric d is a w-distance on X . Many other examples of w-distance are given in [1], [2], [3], [4], [5], [6]. Note that in general for $x, y \in X$, $w(x,y) \ne w(y,x)$.

Example 1.3

If $X = \{\frac{1}{n} \setminus n \in \mathbb{N}\} \cup \{0\}$. For each $x, y \in X$, $d(x,y) = x + y$ if $x \ne y$ and $d(x,y) = 0$ if $x = y$ is a metric on X and (X,d) is a complete metric space. Moreover by defining $p(x,y) = y$, p is a w-distance on (X,d) . We find that $p(x,y) \ne p(y,x)$ for all point except at $x = y$.

Lemma 1.4 [1]

Let X be a metric space with metric d and w be a w-distance on X . Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,\infty)$ converge to 0, and $x, y, z \in X$. Then the following hold:

(a) If $w(x_n, y) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$ in particular, if $w(x,y) = 0$ and $w(x,z) = 0$ then $y = z$. Similarly, if $w(y, x_n) \le \alpha_n$ and $w(z, x_n) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$ in particular, if $w(y,x) = 0$ and $w(z,x) = 0$ then $y = z$.

(b) If $w(x_n, y_n) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z . Similarly, if $w(y_n, x_n) \le \alpha_n$ and $w(z, x_n) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z .

(c) If $w(x_n, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence. Similarly, if $w(x_m, x_n) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.

(d) If $w(y, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence. Similarly, if $w(x_n, y) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.5

We say a multivalued map $T : X \to CL(X)$ is :

(i) Non commutative w-contraction if there exist a w-distance w on X and a constant $h \in (0, \frac{1}{2})$ such that for any $x, y \in X$ and $u \in T(x)$ there is $v \in T(y)$ with $w(u,v) \le h[w(x,y) + w(y,x)]$.

(ii) Generalized non commutative w-contraction if there exists a w-contractive if there exists a w-distance w on X such that for any $x, y \in X$ and $u \in T(x)$ there is $v \in T(y)$ with

$$w(u,v) \le k(w(x,y), w(y,x))[w(x,y) + w(y,x)]$$

Where $k : [0,\infty) \times [0,\infty) \to [0, \frac{1}{2})$

With $\lim_{(r_1,r_2)\to(t_1^+,t_2^+)} \sup k(r_1,r_2) < \frac{1}{2}$ for every $(t_1,t_2) \in [0,\infty) \times [0,\infty)$.

Definition 1.6

A sequence $\{x_n\}$ in X is said to be an orbit of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \ge 1$.

A Fixed Point Result.

First, we prove our key lemma in the setting of metric spaces.

Lemma 2.1

Let $T : X \to CL(X)$ be generalized non commutative w-contraction map. Then there exists an orbit $\{x_n\}$ of T at x_0 such that the sequences of nonnegative numbers $\{w(x_n, x_{n+1})\}$ and $\{w(x_{n+1}, x_n)\}$ are converging to zero and the sequence $\{x_n\}$ is Cauchy.

Proof.

Let x_0 be an arbitrary but fixed element of X and $x_1 \in T(x_0)$. Since T is generalized non commutative w-contraction, there is $x_2 \in T(x_1)$ such that

$$w(x_1,x_2) \le k(w(x_0,x_1), w(x_1,x_0))[w(x_0,x_1) + w(x_1,x_0)]$$

Continuing this process, we get a sequence $\{x_n\}$ in X such that $x_{n+1} \in T(x_n)$ and

$$w(x_n,x_{n+1}) \le k(w(x_{n-1},x_n), w(x_n,x_{n-1})) \times [w(x_{n-1},x_n) + w(x_n,x_{n-1})]$$

Thus for all $n \geq 1$, we have

$$w(x_n, x_{n+1}) < \frac{1}{2}[w(x_{n-1}, x_n) + w(x_n, x_{n-1})]$$

Write $t_n = w(x_n, x_{n+1})$, $t_n^{\backslash} = w(x_{n+1}, x_n)$. Suppose $\lim_{n\to\infty} t_n = \lambda_1 > 0$ and $\lim_{n\to\infty} t_n^{\backslash} = \lambda_2 > 0$. Then we have

$$t_n \leq k(t_{n-1}, t_{n-1}^{\backslash})[t_{n-1} + t_{n-1}^{\backslash}]$$

Now, taking limits as $n \to \infty$ on both sides, we get

$$\lambda_1 \leq \lim_{n\to\infty} \sup k(t_{n-1}, t_{n-1}^{\backslash})(\lambda_1 + \lambda_2)$$

Without loss of generality let $\lambda_2 \leq \lambda_1$, then

$$\lambda_1 \leq \lim_{n\to\infty} \sup k(t_{n-1}, t_{n-1}^{\backslash})(\lambda_1 + \lambda_2) < \lambda_1$$

Which is impossible and hence the sequence of nonnegative numbers $\{t_n\}$ which converges to zero

Similarly with respect to the sequence $\{w(x_{n+1}, x_n)\}$.

Finally, we show that $\{x_n\}$ is a Cauchy sequence.

Let $\alpha = \lim_{(r_1,r_2)\to(0,0)} \sup k(r_1, r_2) < \frac{1}{2}$. Then there exists a real number h such that $\alpha < h < \frac{1}{2}$ and for sufficiently large n we have, $k(t_{n-1}, t_{n-1}^{\backslash}) < h$. Thus for sufficiently large n we have $t_n \leq h[t_{n-1} + t_{n-1}^{\backslash}]$. Consequently,

$$w(x_1, x_2) \leq h[w(x_0, x_1) + w(x_1, x_0)]$$

$$w(x_2, x_3) \leq h[w(x_1, x_2) + w(x_2, x_1)]$$

$$\leq h[h[w(x_0, x_1) + w(x_1, x_0)] + h[w(x_1, x_0) + w(x_0, x_1)]]$$

$$\leq 2h^2[w(x_0, x_1) + w(x_1, x_0)]$$

$$w(x_3, x_4) \leq h[w(x_2, x_3) + w(x_3, x_2)]$$

$$\leq h[2h^2[w(x_0, x_1) + w(x_1, x_0)] + 2h^2[w(x_0, x_1) + w(x_1, x_0)]]$$

$$\leq 4h^3[w(x_0,x_1)+w(x_1,x_0)]$$

$$w(x_4,x_5)\leq h[w(x_3,x_4)+w(x_4,x_3)]$$

$$\leq h[4h^3[w(x_0,x_1)+w(x_1,x_0)]+4h^3[w(x_0,x_1)+w(x_1,x_0)]]$$

$$\leq 8h^4[w(x_0,x_1)+w(x_1,x_0)]$$

And finally we get,

$$w(x_n,x_{n+1})\leq 2^{n-1}h^n[w(x_0,x_1)+w(x_1,x_0)] \quad \forall n=1,2,\ldots$$

Now, for any $n,m\in \mathbb{N}$, $m>n$, we have

$$w(x_n,x_m)\leq w(x_n,x_{n+1})+$$
$$w(x_{n+1},x_{n+2})+\ldots+w(x_{m-1},x_m)$$

$$<2^{n-1}h^n[w(x_0,x_1)+w(x_1,x_0)]+2^n h^{n+1}[w(x_0,x_1)+w(x_1,x_0)]$$

$$+\ldots+2^{m-2}h^{m-1}[w(x_0,x_1)+w(x_1,x_0)]$$

$$<2^{n-1}h^n[w(x_0,x_1)+w(x_1,x_0)][1+2h+4h^2+\ldots+2^{m-n-1}h^{m-n-1}]$$

$$<\frac{1}{2}\left(\frac{(2h)^n}{1-2h}\right)[w(x_0,x_1)+w(x_1,x_0)]$$

Since $2h<1$, then $(2h)^n\to 0$ as $n\to\infty$ and $\{x_n\}$ is a Cauchy sequence. Similarly with respect to the sequence $\{w(x_m,x_n)\}$

Theorem 2.2

Let X be a complete metric space. Then each generalized non commutative w-contraction map $T:X\to CL(X)$ has a fixed point.

Proof.

It follows from the previous lemma 2.1 that there exists an orbit $\{x_n\}$ of T which is a Cauchy sequence and the sequence $w(x_n,x_{n+1})\to 0$. Due to the completeness of X , there exists some $v_0\in X$ such that $\lim_{n\to\infty}x_n=v_0$. Since $w(x_n,.)$ is a lower semicontinuous and $x_m\to v_0\in X$, it follows from lemma 2.1 that

$$w(x_n,v_0)\le \liminf_{m\to\infty} w(x_n,x_m) < \left(\frac{2^{n-1}h^n}{1-2h}\right)[w(x_0,x_1)+w(x_1,x_0)]$$

And

$$w(v_0,x_n)\le \liminf_{m\to\infty} w(x_m,x_n) < \left(\frac{2^{n-1}h^n}{1-2h}\right)[w(x_1,x_0)+w(x_0,x_1)]$$

Which implies $w(x_n,v_0)\to 0$ and $w(v_0,x_n)\to 0$. Now since $x_n\in T(x_{n-1})$ and T is generalized non commutative w-contraction map, there is $u_n\in T(v_0)$ such that

$$w(x_n,u_n)\le k(w(x_{n-1},v_0),w(v_0,x_{n-1}))[w(x_{n-1},v_0)+w(v_0,x_{n-1})]$$
$$<\frac{1}{2}[w(x_{n-1},v_0)+w(v_0,x_{n-1})]\to 0$$

Similarly $w(u_n,x_n)\to 0$. Thus it follows from the condition b in lemma (1.4). That $u_n\to v_0$. Since $T(v_0)$ is closed, we get $v_0\in T(v_0)$.

Definition 2.3

A multivalued map $T:X\to 2^X$ is called a non commutative k_w-map if there exist a non negative number $h\in(0,\frac{1}{4})$ and if $M\subseteq X$, $\forall x,y\in M$ there exist $u\in T(x)$, $v\in T(y)$ such that

$$\max\{w(u,v),w(v,u)\}\le h[w(x,u)+w(u,x)+w(y,v)+w(v,y)]$$

Theorem 2.4

Let $T:M\to CL(M)$ be a non commutative k_w-map such that

$$\inf\{w(x,u)+w(u,x)+w(x,Tx)+w(Tx,x):x\in X\}>0$$

For every $u\in X$ with $u\notin T(u)$. Then T has a fixed point.

Proof .

Let $u_0\in M$ be arbitrary and $u_1\in T(u_0)$ be fixed. Since T is a non commutative k_w-map there exists $u_2\in T(u_1)$ such that

$$\max\{w(u_1,u_2),w(u_2,u_1)\} \le h\{w(u_0,u_1)+w(u_1,u_0)+w(u_1,u_2)+w(u_2,u_1)\}$$

If $\max\{w(u_1,u_2), w(u_2,u_1)\} = w(u_1,u_2)$, then

$$w(u_1,u_2) \le h\{w(u_0,u_1)+w(u_1,u_0)+w(u_1,u_2)+w(u_2,u_1)\}$$

Then

$$(1-h)w(u_1,u_2) \le h\{w(u_0,u_1)+w(u_1,u_0)+w(u_2,u_1)\} \qquad (1)$$

But $w(u_2,u_1) \le h\{w(u_1,u_2)+w(u_2,u_1)+w(u_0,u_1)+w(u_1,u_0)\}$

Then $(1-h)w(u_2,u_1) \le h\{w(u_1,u_2)+w(u_0,u_1)+w(u_1,u_0)\}$

And we get $w(u_2,u_1) \le \dfrac{h}{(1-h)}\{w(u_1,u_2)+w(u_0,u_1)+w(u_1,u_0)\}$ (2)

From inequalities **(1)** and **(2)** we have

$$(1-h)w(u_1,u_2) \le h\{w(u_0,u_1)+w(u_1,u_0)+\frac{h}{(1-h)}\{w(u_1,u_2)+w(u_0,u_1)+w(u_1,u_0)\}\}$$

After some calculations, we get

$$w(u_1,u_2) \le \left(\frac{h}{1-2h}\right)[w(u_0,u_1)+w(u_1,u_0)]$$

But since $0 < \left(\dfrac{h}{1-2h}\right) < \dfrac{1}{2}$, let $k = \left(\dfrac{h}{1-2h}\right)$

So we have $w(u_1,u_2) \le k[w(u_0,u_1)+w(u_1,u_0)]$ (3)

Similarly if $\max\{w(u_1,u_2), w(u_2,u_1)\} = w(u_2,u_1)$,

we get $w(u_2,u_1) \le k[w(u_1,u_0)+w(u_0,u_1)]$
(4)

Also we have

$$\max\{w(u_2,u_3),w(u_3,u_2)\} \le h\{w(u_1,u_2)+w(u_2,u_1)+w(u_2,u_3)+w(u_3,u_2)\}$$

Now if $\max\{w(u_2,u_3), w(u_3,u_2)\} = w(u_2,u_3)$,

then $w(u_2,u_3) \le h\{w(u_1,u_2)+w(u_2,u_1)+w(u_2,u_3)+w(u_3,u_2)\}$

From inequalities (3), (4), we have

$$(1-h)w(u_2,u_3) \le h\{w(u_1,u_2)+w(u_2,u_1)+w(u_3,u_2)\}$$

We get $(1-h)w(u_2,u_3) \le h\{2k[w(u_0,u_1)+w(u_1,u_0)]+w(u_3,u_2)\}$

Also, we have

$$w(u_3,u_2) \le h\{w(u_2,u_3) + w(u_3,u_2) + w(u_1,u_2) + w(u_2,u_1)\}$$

Then $(1-h)w(u_3,u_2) \le h\{w(u_2,u_3) + 2k[w(u_0,u_1) + w(u_1,u_0)]\}$

From which $w(u_3,u_2) \le \frac{h}{(1-h)}\{w(u_2,u_3) + 2k[w(u_0,u_1) + w(u_1,u_0)]\}$

So

$$(1-h)w(u_2,u_3) \le h\{2k[w(u_0,u_1) + w(u_1,u_0)] + w(u_3,u_2)\}$$

$$\le h\{2k[w(u_0,u_1) + w(u_1,u_0)] + \left(\frac{h}{1-h}\right)\{w(u_2,u_3) + 2k[w(u_0,u_1) + w(u_1,u_0)]\}\}$$

Then

$$\left(\frac{1-2h}{1-h}\right)w(u_2,u_3) \le h\{2k[w(u_0,u_1) + w(u_1,u_0)] + \left(\frac{2kh}{1-h}\right)[w(u_0,u_1) + w(u_1,u_0)]\}$$

So, we get

$$w(u_2,u_3) \le \left(\frac{1-h}{1-2h}\right)\left(\frac{2kh}{1-h}\right)[w(u_0,u_1) + w(u_1,u_0)]$$

$$w(u_2,u_3) \le \left(\frac{2kh}{1-2h}\right)[w(u_0,u_1) + w(u_1,u_0)]$$

$$w(u_2,u_3) \le 2k^2[w(u_0,u_1) + w(u_1,u_0)]$$

Similarly if $\max\{w(u_2,u_3), w(u_3,u_2)\} = w(u_3,u_2)$, then

$w(u_3,u_2) \le 2k^2[w(u_1,u_0) + w(u_0,u_1)]$

Also, we have

$$w(u_3,u_4) \le h\{w(u_2,u_3) + w(u_3,u_2) + w(u_3,u_4) + w(u_4,u_3)\}$$

From which

$$(1-h)w(u_3,u_4) \le h\{4k^2[w(u_0,u_1) + w(u_1,u_0)] + w(u_4,u_3)\}$$

But

$$w(u_4,u_3) \le \left(\frac{h}{1-h}\right)\{w(u_3,u_4) + 4k^2[w(u_0,u_1) + w(u_1,u_0)]\}$$

Then

$$(1-h)w(u_3,u_4) \le h\{4k^2[w(u_0,u_1)+w(u_1 u_0)]+ \left(\frac{h}{1-h}\right)\{w(u_3,u_4)+4k^2[w(u_0,u_1)+w(u_1 u_0)]\}\}$$

$$\left(\frac{1-2h}{1-h}\right)w(u_3,u_4) \le \left(\frac{4k^2h}{1-h}\right)[w(u_0,u_1)+w(u_1,u_0)]$$

Then

$$w(u_3,u_4) \le 4k^3[w(u_0,u_1)+w(u_1,u_0)]$$

By induction, we get $w(u_n,u_{n+1}) \le 2^{n-1}k^n[w(u_0,u_1)+w(u_1,u_0)]$

To prove that the sequence $\{u_n\}$ is a Cauchy sequence, let $m > n$ arbitrary, then

$$w(u_n,u_m) \le w(u_n,u_{n+1})+w(u_{n+1},u_{n+2})+...+w(u_{m-1},u_m)$$

$$\le [2^{n-1}k^n + 2^n k^{n+1} + ... + 2^{m-2}k^{m-1}][w(u_0,u_1)+w(u_1,u_0)]$$

$$\le 2^{n-1}k^n[1+2k+...+2^{m-n-1}k^{m-n-1}][w(u_0,u_1)+w(u_1,u_0)]$$

$$\le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_0,u_1)+w(u_1,u_0)]$$

Similarly

$$w(u_m,u_n) \le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_1,u_0)+w(u_0,u_1)]$$

So $\{u_n\}$ is a Cauchy sequence. Since X is complete, then there exists $v_0 \in X$ such that $u_n \to v_0 \in X$, M being closed we have $v_0 \in M$. Let $n \in \mathbb{N}$ be fixed. Since $u_m \to v_0$ and $w(u_n,.)$ is a lower semicontinuous, we get

$$w(u_n, v_0) \le \lim_{m\to\infty} \inf w(u_n, u_m) \le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_0, u_1) + w(u_1, u_0)]$$

Similarly

$$w(v_0, u_n) \le \lim_{m\to\infty} \inf w(u_m, u_n) \le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_1, u_0) + w(u_0, u_1)]$$

So $w(u_n, v_0) \to 0$ and $w(v_0, u_n) \to 0$ as $n \to \infty$.

Now, we want to prove that $v_0 \in T(v_0)$ by contradiction. Assume that $v_0 \notin T(v_0)$. Then by hypothesis we have

$$0 < \inf\{w(u, v_0) + w(v_0, u) + w(u, Tu) + w(Tu, u) : u \in X\}$$

$$\le \inf\{w(u_n, v_0) + w(v_0, u_n) + w(u_n, Tu_n) + w(Tu_n, u_n) : n \in \mathbb{N}\}$$

$$\le \inf\{w(u_n, v_0) + w(v_0, u_n) + w(u_n, u_{n+1}) + w(u_{n+1}, u_n) : n \in \mathbb{N}\}$$

$$\le \inf\{\left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_0, u_1) + w(u_1, u_0)] + \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_1, u_0) + w(u_0, u_1)] +$$

$$2^{n-1}k^n[w(u_0, u_1) + w(u_1, u_0)] + 2^{n-1}k^n[w(u_1, u_0) + w(u_0, u_1)]\} = 0$$

Which is impossible and hence $v_0 \in T(v_0)$.

Theorem 2.5

Each non commutative k_w-multivalued map $T : M \to CL(M)$ has a fixed point provided that for any iterative sequence $\{u_n\}$ in M with $u_n \to v_0 \in M$. The sequence of real numbers $\{w(v_0, u_n)\}$ and $\{w(u_n, v_0)\}$ converges to zero.

Proof .

From the proof of the previous theorem. There exists a convergent iterative sequence $\{u_n\}$ such that $u_n \to v_0 \in M$ with

$$w(u_n,v_0)\le \liminf_{m\to\infty} w(u_n,u_m)\le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_0,u_1)+w(u_1,u_0)]$$

Similarly

$$w(v_0,u_n)\le \liminf_{m\to\infty} w(u_m,u_n)\le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_1,u_0)+w(u_0,u_1)]$$

And

$$w(u_n,u_{n+1})\le 2^{n-1}k^n[w(u_0,u_1)+w(u_1,u_0)]$$

Similarly

$$w(u_{n+1},u_n)\le 2^{n-1}k^n[w(u_1,u_0)+w(u_0,u_1)]$$

Where

$k=\left(\frac{h}{1-2h}\right)<\frac{1}{2}$. Note that $w(u_n,v_0)\underset{n\to\infty}{\to}0$ and $w(v_0,u_n)\underset{n\to\infty}{\to}0$. Further, since $u_n\in T(u_{n-1})$ and T is a non commutative k_w-multivalued-map, there is $v_n\in T(v_0)$ such that

$$\max\{w(u_n,v_n),w(v_n,u_n)\}\le h\{w(u_{n-1},u_n)+w(u_n,u_{n-1})+w(v_0,v_n)+w(v_n,v_0)\}$$

If $\max\{w(u_n,v_n),w(v_n,u_n)\}=w(u_n,v_n)$, then

$$w(u_n,v_n)\le h\{w(u_{n-1},u_n)+w(u_n,u_{n-1})+w(v_0,v_n)+w(v_n,v_0)\}$$

$$\le h\{w(u_{n-1},u_n)+w(u_n,u_{n-1})+w(v_0,u_n)+w(u_n,v_n)+w(v_n,u_n)+w(u_n,v_0)\}$$

$$(1-h)w(u_n,v_n)\le h\{w(u_{n-1},u_n)+w(u_n,u_{n-1})+w(v_0,u_n)+w(v_n,u_n)+w(u_n,v_0)\}$$

But

$$w(v_n,u_n)\le h\{w(v_0,v_n)+w(v_n,v_0)+w(u_{n-1},u_n)+w(u_n,u_{n-1})\}$$

$$\le h\{w(v_0,u_n)+w(u_n,v_n)+w(v_n,u_n)+w(u_n,v_0)+w(u_{n-1},u_n)+w(u_n,u_{n-1})\}$$

$$w(v_n,u_n)\le\left(\frac{h}{1-h}\right)\{w(v_0,u_n)+w(u_n,v_n)+w(u_n,v_0)+w(u_{n-1},u_n)+w(u_n,u_{n-1})\}$$

$$w(u_n,v_n)\le\left(\frac{h}{1-2h}\right)\{w(u_{n-1},u_n)+w(u_n,u_{n-1})+w(v_0,u_n)+w(u_n,v_0)\}\to 0$$

Similarly if $\max\{w(u_n,v_n),w(v_n,u_n)\}=w(v_n,u_n)$, then

$$w(v_n,u_n)\le\left(\frac{h}{1-2h}\right)\{w(u_n,v_0)+w(v_0,u_n)+w(u_n,u_{n-1})+w(u_{n-1},u_n)\}\underset{n\to\infty}{\longrightarrow}0$$

Thus $w(u_n,v_n)\underset{n\to\infty}{\longrightarrow}0$ and $w(u_n,v_0)\underset{n\to\infty}{\longrightarrow}0$, from lemma 1.4 (b), we get $v_n\to v_0$

Similarly for $w(v_n,u_n)\underset{n\to\infty}{\longrightarrow}0$ and $w(v_0,u_n)\underset{n\to\infty}{\longrightarrow}0$ which implies $v_n\to v_0$

Since $v_n\in T(v_0)$ which is closed, then $v_0\in T(v_0)$.

Now, we prove the existence of a common fixed point for non commutative k_w-multivalued map.

Theorem 2.6

Let $\{T_n\}$ be a sequence of multivalued maps of M into $CL(M)$. Suppose that there exists a constant $0<h<\frac{1}{4}$ such that for any $T_i,T_j\in\{T_n\}$ and for any $x\in M$, $u\in T_i(x)$. There exists $v\in T_j(y)$ for all $y\in M$ with

$$\max\{w(u,v),w(v,u)\}\le h\{w(x,u)+w(u,x)+w(y,v)+w(v,y)\}$$

And for each $n\ge1$

$$\inf\{w(x,u)+w(u,x)+w(x,T_n(x))+w(T_n(x),x):x\in X\}>0 .$$

For any $u\notin T_n(u)$. Then $\{T_n\}$ has a common fixed point.

Proof .

Let u_0 be an arbitrary element of M and let $u_1\in T_1(u_0)$. Then there is $u_2\in T_2(u_1)$, such that

$$4\,w(u_1,u_2)\le\left(\frac{h}{1-2h}\right)[w(u_0,u_1)+w(u_1,u_0)]$$

But $0<\frac{h}{1-2h}<\frac{1}{2}$

Then put $k = \dfrac{h}{1-2h}$ and we get $w(u_1,u_2) \le k[w(u_0,u_1)+w(u_1,u_0)]$

Similarly

$$w(u_2,u_1) \le k[w(u_1,u_0)+w(u_0,u_1)]$$

So there exists a sequence $\{u_n\}$ such that $u_{n+1} \in T_{n+1}(u_n)$ and for all $n \ge 1$.

From theorem (2.4)

$$w(u_n,u_{n+1}) \le 2^{n-1}k^n[w(u_0,u_1)+w(u_1,u_0)]$$

And

$$w(u_{n+1},u_n) \le 2^{n-1}k^n[w(u_1,u_0)+w(u_0,u_1)]$$

for all $n \ge 1$. Then $\{u_n\}$ is a Cauchy sequence in X .

Let $\lim_{n\to\infty} u_n = p \in M$. We show that $p \in \bigcap_{n\ge1} T_n(p)$. Let T_m be an arbitrary member of $\{T_n\}$. Since $u_n \in T_n(u_{n-1})$, by hypothesis there is $s_n \in T_m(p)$ such that

$$w(u_n,s_n) \le \left(\frac{h}{1-2h}\right)\{w(u_{n-1},u_n)+w(u_n,u_{n-1})+w(p,s_n)+w(s_n,p)\}$$

And

$$w(s_n,u_n) \le \left(\frac{h}{1-2h}\right)\{w(s_n,p)+w(p,s_n)+w(u_n,u_{n-1})+w(u_{n-1},u_n)\}$$

We proceed as in the proof of Theorem 3.2 and we get

$$w(u_n,p) \le \liminf_{m\to\infty} w(u_n,u_m) \le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_0,u_1)+w(u_1,u_0)]$$

And

$$w(p,u_n) \le \liminf_{m\to\infty} w(u_m,u_n) \le \left(\frac{2^{n-1}k^n}{1-2k}\right)[w(u_1,u_0)+w(u_0,u_1)]$$

Now assume that $p \notin T_m(p)$. Then, by hypothesis and for $n > m$ and $m \ge 1$ we have

$$0 < \inf\{w(u,p)+w(p,u)+w(u,T_m u)+w(T_m u,u) : u \in X\}$$

$$\le \inf\{w(u_{m-1},p)+w(p,u_{m-1})+w(u_{m-1},T_m(u_{m-1}))+w(T_m(u_{m-1}),u_{m-1}): m\in \mathbb{N}\}$$

$$\le \inf\{w(u_{m-1},p)+w(p,u_{m-1})+w(u_{m-1},u_m)+w(u_m,u_{m-1}): m\in \mathbb{N}\}$$

$$\le \inf\{\left(\frac{2^{m-2}k^{m-1}}{1-2k}\right)[w(u_0,u_1)+w(u_1,u_0)]+\left(\frac{2^{m-2}k^{m-1}}{1-2k}\right)[w(u_1,u_0)+w(u_0,u_1)]$$

$$+2^{m-2}k^{m-1}[w(u_0,u_1)+w(u_1,u_0)]+2^{m-2}k^{m-1}[w(u_1,u_0)+w(u_0,u_1)]\}=0$$

Which is impossible and hence $p\in T_m(p)$. But T_m is an arbitrary; hence p is a common fixed point.

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References

[1] O. Kada, T. Susuki and W. Takahashi. “Nonconvex minimization theorems and fixed point theorems in complete metric spaces”. Mathematica Japonica., 44 (1996), 381-391.

[2] T. Suzuki and W. Takahashi, “Fixed point Theorems and Characterizations of metric completeness”, Topological Methods in Nonlinear Analysis., 8 (1996), 371- 382.

[3] W. Takahashi. “Existence theorems in metric spaces and characterizations of metric completeness”. Josai Mathematical Monographs, 1 (1999). 67-85.

[4] W. Takahashi, “Nonlinear Functional Analysis”; Fixed point theory and its applications, Yokohama Publishers, 2000.

[5] Sushanta Kumar Mohanta, Generalized *W*-Distance and a Fixed Point Theorem, Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 18, 853 – 860.

[6] Sushanta Kumar Mohanta, A Fixed Point Theorem Via Generalized *W*-Distance, Bulletin of Mathematical Analysis and Applications, Volume 3 Issue 2(2011), Pages 134-139.