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# Some Degree Conditions on triple vertices for Digraph to be Supereulerian

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**Abstract**

A digraph D is supereulerian if D has a spanning eulerian subdigraph. We prove that a strong digraph D of order $n \geq 4$ satisfies the following conditions: for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent, if there is no arc from x to z, then $d(x) + d(y) + d^{+}(x) + d^{-}(z) \geq 3n - 5$. Then D is supereulerian.

**Key words**. strong arc connectivity, eulerian digraphs, supereulerian digraphs.

## 1 Introduction

Suppose that D is finite and simple digraphs (without loops or parallel arcs, but possibly with cycles of length two). Denote by V (D), A(D) its vertex set and arc set, respectively, when D is clear from context we simply write V and A. Undefined terms and notations will follow [5] and [7]. Throughout this paper, we use the notation (u, v) to denote an arc oriented from u to v in a digraph and the notation [u, v] to denote an edge between u and v. When $(u, v) \in A(D)$, we say that u and v are adjacent. For integers $n, m > 0$; we use $K_{n,m}$ to denote the complete bipartite graph.

For digraphs H and D, by $H \subseteq D$ we mean that H is a subdigraph of D. Following [5], for a digraph D with $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x; y) \in A(D) : x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial^{+}_{D}(X) = (X, V(D) - X)_D \text{ and } \partial^{-}_{D}(X) = (V(D) - X, X)_D.$$

For a vertex $v \in V(D)$, $d^{+}_{D}(v) = |\partial^{+}_{D}(\{v\})|$ and $d^{-}_{D}(v) = |\partial^{-}_{D}(\{v\})|$ are the out-degree and the in-degree of v in D, respectively. $d_D(v) = d^{+}_{D}(v) + d^{-}_{D}(v)$.

A digraph D is complete if, for every pair x, y of distinct vertices of D both (x, y) and (y, x) are in D. A digraph D is called strong if there is a dipath from x to y and a dipath from y to x for all x, y $\in$ V (D). Given an (x, y)-dipath P, we denote by P ]x, y[ the dipath P- {x, y}. Given a dipath P = $v_1, v_2, \ldots, v_k$ we denote by P [$v_i$, $v_j$], where $1 \le i < j \le k$ the subdipath of P starts at $v_i$ and ends at $v_j$. For a subdigraph H of a digraph D, an (x, y)-dipath P is an (H, H)-dipath if x, y $\in$ V (H) and V (P ) $\cap$ V (H) = {x, y}. We say that an ordered pair of vertices (x, y) is dominated (dominating) if there exists z $\in$ V (D), with (z, x), (z, y) $\in$ A(D) ((x, z), (y, z) $\in$ A(D)).

A walk in D is an alternating sequence W = $x_1a_1x_2a_2 \ldots x_{k-1}a_{k-1}x_k$ of vertices $x_i$ and arcs $a_j$ from D such that $a_i = x_ix_{i+1}$ for i = 1,…,k-1. A walk is closed if $x_1 = x_k$, and open otherwise. If all the arcs of a walk are distinct we call it a ditrail. If a ditrail starts at s and ends at t we call it (s, t)-ditrail.

Let H be a subdigraph of a digraph D and X $\subseteq$ A(D). Then we use H + X to denote the subdigraph D[A(H) $\cup$ X] induced by A(H) $\cup$ X. If H' is also a subdigraph of D, then we use H + H' for H + A(H').

Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [1] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs, and they in [1] indicated that this problem would be very difficult. Pulleyblank [11] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic.

It is natural to consider the supereulerian problem in digraphs. A digraph D is eulerian if D is connected and for any v $\in$ V (D), $d^+_D(v) = d^-_D(v)$. A digraph D is supereulerian if D contains a spanning eulerian subdigraph. A digraph D is a closed ditrail if it is eulerian. The main problem is to determine supereulerian digraphs.

Several efforts in supereulerian digraphs have been made. However, contrary to the case of undirected graphs, not much work has been done yet for supereulerian digraphs. The earlier studies were done by Gutin ([2, 3]). For what has been recently done see [6],[12],[9] and [10].

The property of being supereulerian is at the same time relaxation of being hamiltonian: being supereulerian digraph means having a closed ditrail covering all the vertices of the digraph; being hamiltonian means having a closed ditrail covering all vertices of the digraph without using a vertex twice. In this paper we analyze some sufficient conditions for a digraph to be supereulerian.

The purpose of the following section is to show that, as it is the case for undirected graphs, some sufficient degree conditions for hamiltonicity in digraphs can be (slightly) weakened to become sharp sufficient conditions for supereulerianity. The next well known theorem in hamiltonian digraphs is due to Meyniel.

**Theorem 1.1** (Meyniel [4]) A strong digraph D on n vertices satisfying $d(x) + d(y) \ge 2n - 1$ for all pairs of non-adjacent vertices x, y is hamiltonian.

Bang-Jensen and Maddaloni in [6] proved the following theorem which is a similar result to Meyniel's theorem for supereulerian digraphs.

**Theorem 1.2** (J. Bang-Jensen and A. Maddaloni [6]) Let D be a strong simple digraph on n vertices. If $d(x) + d(y) \ge 2n - 3$ for any pair of non-adjacent vertices x and y, then D is supereulerian.

The next theorem is due to Y. Manoussakis.

**Theorem 1.3** (Y. Manoussakis [13]) Suppose that a strong digraph D of order $n \geq 2$ satisfies the following condition: for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent, if there is no arc from x to z, then

$$d(x) + d(y) + d^{+}(x) + d^{-}(z) \geq 3n - 2.$$

Then D is hamiltonian.

We will study the case in which the condition in theorem 1.3 is sufficient for a digraph to be supereulerian.

We will use the following lemma later as a necessary condition for a digraph to be supereulerian.

**Lemma 1.1** (K.A. Alsatami et al, Lemma 2 of [8]) A digraph D is nonsupereulerian if there exist vertex-disjoint subdigraphs $\{A, B_1, \ldots, B_m\}$ of D, for some integer $m > 0$, satisfying each of the following.

(i) $N^{-}(B_i) \subseteq V(A)$, $\forall i \in \{1, 2, \ldots, m\}$.

(ii) $|\partial^{-}(A)| \leq m - 1$.

## 2 Main Theorem

**Defintion 2.1** Let D be a strong digraph and G be a maximal eulerian subdigraph with respect to vertices where $|V(G)|<|V(D)|$. If P is a dipath in G and G-P is not connected then, we say that D is semi-max-digraph.

**Theorem 2.1** Suppose that a strong digraph D of order $n \geq 4$ satisfies the following condition: for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent. If there is no arc from x to z, then

$$d(x) + d(y) + d^{+}(x) + d^{-}(z) \geq 3n - 5.$$

Then D is supereulerian or semi-max-digraph.

**Proof** Since D is strong, D must have an eulerian subdigraph. Let $\{S_i\}_{i\geq 1}$ where $i \in N$, be the set of eulerian subdigraphs of D such that among all eulerian subdigraphs of D

$$\text{Therefor let } |V(S_i)| \text{ be maximized.} \quad (1)$$

Let S be an eulerian subdigraph of $\{S_i\}$ such that among all eulerian subdigraphs of $\{S_i\}$

$$\text{Therefor let } |A(S)| \text{ be maximized.} \quad (2)$$

If $|V(S)| = |V(D)|$, then S is a spanning eulerian subdigraph of D and we are done. Assume by contradiction that $|V(D)| > |V(S)| > 1$. Hence $V(D) - V(S) \neq \phi$. Since D is strong, there exists an (S, S)-dipath Q on at least three vertices. Let Q be chosen so that:

the length of a shortest dipath P in S between the endpoints of Q is minimized. (3)

Assume that $V(Q) \cap V(S) = \{z, r\}$, where $z; r$ are the first and the last vertex of Q. Assume that x is the first vertex of $Q]z, r[$, namely, $(z, x) \in A(Q)$. If $P = (z, r)$, then $S - (z, r) + Q$ is an eulerian subdigraph with at least one more vertex than S, contrary to (1), moreover, by the maximality of S, z cannot equal r. Therefore, $|V(P)| \geq 3$ and $|V(Q)| \geq 3$. Assume that $y_1$ and $y_d$ are the first and the last vertex of $P\,]z, r[$, namely, $\{(z, y_1), (y_d, r)\} \subseteq A(P)$. Note that $y_1$ may equal $y_d$. There exists a vertex $y_c \in V(P\,]z; r[)$ where $1 \leq c \leq d$ such that

$$|\partial^+_S(y_c) \cap A(S)| = 1 \text{ and } |\partial^-_S(y_c) \cap A(S)| = 1, \quad (4)$$

otherwise $S - A(P) + Q$ is a greater closed ditrail, contradiction with maximality of $V(S)$.

**(A) $|V(P)| \geq 4$** (when the number of vertices of P is greater than or equal 4):

Let $M = \{y_1, y_2, \ldots, y_d\}$ be the vertices of $P\,]z, r[$ where $|M| = d$. Let $|V(S)| = s$.
Then $d_M(x) = 0$, (by minimality of P).
And $d_{V(S) - M}(x) \leq s - d$,

this because any arc increase on this number leads to the existence of a vertex $x_s \in V(S) - M$ such that $\{(x, x_s), (x_s, x)\} \subseteq A(D)$ and we get a greater closed ditrail which is a contradiction with maximality of $V(S)$.

(Here we consider that all arcs from x to $V(S) - (M \cup \{z\})$ are out arcs to maximize $d^+(x)$). (5)

So,

$$d_{V(S)}(x) = d_M(x) + d_{V(S) - M}(x) \leq s - d:$$

$d^+_{V(S)}(x)$ takes its maximum value when all arcs between x and $V(S)$ are from x to $V(S) - M$ except the arc $(z, x)$, so by (5) $d^+_{V(S)}(x) \leq s - d - 1$.

We have $d_M(y_c) \leq d + 1$:

the reasons are the following:

**(i)** If $y_i \in M$ and $y_i$ is not a neighbor of $y_c$ in P then

$$|\{(y_c, y_i), (y_i, y_c)\} \cap A(D)| \leq 1, \quad (6)$$

otherwise $\{(y_c, y_i), (y_i, y_c)\} \subseteq A(D) - A(S)$ (we used 4) and $S + \{(y_c, y_i), (y_i, y_c)\}$ is a greater closed ditrail with respect to arcs which is a contradiction with maximality of $A(S)$.
By the previous inequality labeled with number (6) and for our purpose we choose

$$\{(y_{c+1+j}, y_c), (y_c, y_{c-1-k})\} \subseteq A(D), \quad (7)$$

and

$$\{(y_c, y_{c+1+j}), (y_{c-1-k}, y_c)\} \cap A(D) = \phi \quad \forall\ 1 \leq j \leq d - c - 1 \text{ and } 1 \leq k \leq c - 2.$$

**(ii)** We have
$|(y_{c-1}, y_c), (y_c, y_{c-1}), (y_c, y_{c+1}), (y_{c+1}, y_c) \cap A(D)| \leq 4.$
By (i) and (ii) we have $d_M(y_c) \leq d + 1$.

We have $d_{V(S)-M}(y_c) \leq s - d - 1$:

the reasons are the following:

**(i)** By using the equalities labeled with number (4) then

$$|(y_c, f), (f, y_c)| \leq 1 \;\; \forall\, f \in V(S) - V(P), \tag{8}$$

otherwise $A(S) \cup \{(y_c, f), (f, y_c)\}$ is a greater closed ditrail with respect to arcs (contradiction with maximality of A(S)).

**(ii) case.(1)** $d \geq 3$ and $2 \leq c \leq d - 1$,

$$\{(r, y_c), (y_c, r), (y_c, z), (z, y_c)\} \cap A(D) = \phi \tag{9}$$

If $(r, y_c) \in A(D)$ then by the containment labeled with number (7) we have $S - (z, y_1)+Q+(r, y_c)+(y_c, y_1)$ as a greater closed ditrail (contradiction with maximality of V (S)). If $(y_c, r) \in A(D)$ then by the containment number (7) we have $S - (y_d, r) + (y_d, y_c) + (y_c, r)$ as a greater closed ditrail with respect to arcs (contradiction with maximality of A(S)). If $(y_c, z) \in A(D)$ then by the containment number (7) we have $S - (y_d, r) + (y_d, y_c) + (y_c, z) + Q$ as a greater closed ditrail (contradiction with maximality of V (S)). If $(z, y_c) \in A(D)$ then by the containment number (7) we have $S - (z, y_1) + (z, y_c) + (y_c, y_1)$ as a greater ditrail with respect to arcs (contradiction with maximality of A(S)).

Hence by the intersection number (9) and the inquality number (8) $d_{V(S) - M}(y_c) \leq s - d - 2$.

**case.(2)** $d \geq 2$ and $c = 1$,

$$\{(r, y_c), (y_c, r), (y_c, z), (z, y_c)\} \cap A(D) = \{(z, y_c)\} \tag{10}$$

**(1)** $S - (z, y_c) + Q + (r, y_c)$ violates (1) $\Rightarrow (r, y_c) \notin A(D)$.

**(2)** **(a)** $d > 2$,

$|\{(y_d, y_c), (y_{d-1}, y_c), (y_c, r), (y_c, z)\}| \leq 2$, otherwise contradicts maximality of V (S) or maximality of A(S). By the containment number (7) we have $\{(y_d, y_c), (y_{d-1}, y_c)\} \subseteq A(D)$, so when $d > 2$ we have $\{(y_c, r), (y_c, z)\} \cap A(D) = \phi$

**(b)** $d = 2$,

$|\{(y_c, r), (y_c, z), (y_d, y_c)\}| \leq 1$, otherwise contradicts maximality of V (S) or maximality of A(S). By the containment number (7) we have $(y_d, y_c) \in A(D)$, so when $d = 2$ we have $\{(y_c, r), (y_c, z)\} \cap A(D) = \phi$.

Hence by the inequality number (8) and the intersection number (10) $d_{V(S) - M}(y_c) \leq s - d - 1$.

**case.(3)** $d \geq 2$ and $c = d$,

$$\{(r, y_c), (y_c, r), (y_c, z), (z, y_c)\} \cap A(D) = \{(y_c, r)\} \tag{11}$$

**(1)** $S - (y_c, r) + (y_c, z) + Q$ violates maximality of V (S) $\Rightarrow (y_c, z) \notin A(D)$.

**(2)** **(a)** $d > 2$,

$|\{(y_c, y_1), (y_c, y_2), (r, y_c), (z, y_c)\}| \leq 2$, otherwise contradicts maximality of V (S) or maximality of A(S). By the containment number (7) we have $\{(y_c, y_1), (y_c, y_2)\} \subseteq A(D)$, so when $d > 2$ we have $\{(r, y_c), (z, y_c)\} \cap A(D) = \phi$.

**(b)** $d = 2$,

$(r, y_c) \notin A(D)$, otherwise $S - \{(z, y_1), (y_1, y_c)\} + Q + (r, y_c)$ violates maximality of A(S).

$|\{(z, y_c), (y_c, y_1)\}| \leq 1$, otherwise contradicts maximality of A(S). By the containment number (7) we have $(y_c, y_1) \in A(D)$, so when $d = 2$ we have $\{(r, y_c), (z, y_c)\} \cap A(D) = \phi$.

Hence by the inequality number (8) and the intersection number (11) $d_{V(S) - M}(y_c) \leq s - d - 1$.

In the following we will discuss the value of $d^-_{V(S)}(z)$:

**(i)** $|(V(S), z)_D \cap A(S)| = 1$, otherwise $|(z, V(S))_D \cap A(S)| \geq 2$ and there exists a vertex $\bar{a} \in V(S) - M$ such that $(z, \bar{a}) \in A(S)$ ($(z, M - \{y_1\}) \cap A(S) = \phi$ by manimality of P ) which leads to that $S - (z, \bar{a}) + (z, x) + (x, \bar{a})$ is a greater closed ditrail, contradiction with maximality of V (S). Moreover, $(V(S) - M, z) \cap (A(D) - A(S)) = \phi$, otherwise there exists a vertex $\bar{\bar{e}} \in V(S) - M$ such that $(\bar{\bar{e}}, z) \in A(D) - A(S)$ which implies that $S + \{(\bar{\bar{e}}, \bar{\bar{z}}), (z, x), (x, \bar{\bar{e}})\}$ is a greater closed ditrail, contradiction with maximality of V (S).

**(ii)** $\{(y_d, z), (y_{d-1}, z), (y_c, z)\} \cap (A(D) - A(S)) = \phi$, otherwise $S - (y_d, r) + (y_d, z) + Q$ is a greater closed ditrail (violates maximality of V (S)), $S - \{(y_{d-1}, y_d), (y_d, r)\} + (y_{d-1}, z) + Q$ is a greater closed ditrail (violates maximality of A(S)). By the containment number (7) we have $S - (y_d, r) + \{(y_d, y_c), (y_c, z)\} + Q$ as a greater closed ditrail (violates maximality of V (S)).

**(iii)** For any vertex $y_i \in M$ where $i \leq c - 1$ we have $(y_i, z) \notin A(D) - A(S)$, otherwise we have (by using the containment number (7)) $S - (y_d, r) + \{(y_d, y_c), (y_c, y_i), (y_i, z)\} + Q$ is a greater closed ditrail (violates maximality of V (S)). So consider $y_c$ is the first vertex in P ]z, r[ to reach the maximum number of arcs from M to z.

**case.(1)** $d > 2$,

**(1)** $1 \leq c \leq d - 2$,

(i),(ii) and (iii) $\Rightarrow d^-_M(z) + d^-_{V(S)-M}(z) \leq d - 3 - (c - 1) + 1 = d - c - 1$.

**(2)** $d - 1 \leq c \leq d$,

(i),(ii) and (iii) $\Rightarrow d^-_M(z) + d^-_{V(S)-M}(z) \leq 0 + 1 = 1$.

**case.(2)** $d = 2$,

(i),(ii) and (iii) $\Rightarrow d^-_M(z) + d^-_{V(S)-M}(z) \leq 0 + 1 = 1$.

Thus we have

**(a)** $d = 2$ :

$d_{V(S)}(y_c) + d^-_{V(S)}(z) = d_{V(S)-M}(y_c) + d_M(y_c) + d^-_M(z) + d^-_{V(S)-M}(z) \leq s + 1.$

**(b)** $d \geq 3$ and $c = 1$ :

$d_{V(S)}(y_c) + d^-_{V(S)}(z) = d_{V(S)-M}(y_c) + d_M(y_c) + d^-_M(z) + d^-_{V(S)-M}(z) \leq s + 1.$

**(c)** $d \geq 3$ and $c = d$ :

$d_{V(S)}(y_c) + d^-_{V(S)}(z) = d_{V(S)-M}(y_c) + d_M(y_c) + d^-_M(z) + d^-_{V(S)-M}(z) \leq s + 1.$

**(d)** $d \geq 3$ and $2 \leq c \leq d - 2$ :

$d_{V(S)}(y_c) + d^-_{V(S)}(z) = d_{V(S)-M}(y_c) + d_M(y_c) + d^-_M(z) + d^-_{V(S)-M}(z) \leq s.$

**(e)** $d \geq 3$ and $c = d - 1$ :

$d_{V(S)}(y_c) + d^-_{V(S)}(z) = d_{V(S)-M}(y_c) + d_M(y_c) + d^-_M(z) + d^-_{V(S)-M}(z) \leq s.$

By choosing the lowest bound s we conclude that:

$d_{V(S)}(x) + d^+_{V(S)}(x) + d_{V(S)}(y_c) + d^-_{V(S)}(z) \leq s - d + s - d - 1 + s = 3s - 2d - 1.$

Let $H = V(D) - V(S)$, here:

$d_H(x) \leq 2(n - s - 1)$ and $d^+_H(x) \leq n - s - 1$.

$\{(y_c, h), (h, y_c)\} \cap A(D) = \phi \ \forall \ h \in H$, otherwise:

If $h \in V(Q)$, then it contradicts with manimality of P . If $h \notin V(Q)$, then $y_chx + Q - (z, x)$ contradicts with manimality of P and $zxhy_c$ contradicts with manimality of P .

Hence $d_H(y_c) = 0$.

Moreover $d^-_H(z) = 0$ because zxhz + S is a greater closed ditrail (violates maximality of V (S)). Thus we have

$$d_H(x) + d_H(y_c) + d^+_H(x) + d^-_H(z) \leq 2(n - s - 1) + 0 + n - s - 1 + 0.$$

Finally,

$$d(x) + d(y_c) + d^+(x) + d^-(z) \leq 3s - 2d - 1 + 3n - 3s - 3 = 3n - 2d - 4.$$

Since $d \geq 2$, then the right hand side takes its maximum value when d = 2. So the digraph is supereulerian when:

$$d(x) + d(y_c) + d^+(x) + d^-(z) \geq 3n - 1 - 6 = 3n - 7$$

**(B)** $|V(P)| = 3$; namely $M = \{y_c\}$:

**(i)** $d_{V(S)}(x) \leq s - 1$ [by the same way in (A)]

$d_{V(S)}(y_c) \leq s - 1$ [by the same way in (A)]

$d^+_{V(S)}(x) \leq s - 2$ [by the same way in (A)]

$d^-_{V(S)}(z) = 1$, [if there exists a vertex $u \in V(S)$ such that $(u, z) \in A(D) - A(S)$, then $\{(u, z), (z, x), (x, u)\}$ + S is a greater closed ditrail. If there exist two vertices $u, v \in V(S)$ such that $\{(u, z), (v, z)\} \subseteq A(S)$, then there exists a vertex $l \in V(S) - \{y_c\}$ such that $(z, l) \in A(S)$ and $S - (z, l) + \{(z, x), (x, l)\}$ is a greater closed ditrail.]

**(ii)** $d_H(x) \leq n - s - 1$. [Here we do not duplicate $n - s - 1 = |H| - 1 = |V(D) - (V(S) \cup \{x\})|$ because $|M| = 1$, so we just count the arcs leaving x, otherwise, for a vertex $h \in H$ then zxhxr + S - $zy_cr$ contradicts maximality of V (S)].

$d_H(y_c) \leq n - s - 1$. [We just count the arcs leaving $y_c$, otherwise contradicts maximality of V (S) and manimality of P ].

$d^+_H(x) \leq n - s - 1$.

$d^-_H(z) = 0$. [Otherwise contradicts maximality of V (S)

By using (i) and (ii) we have the following:

$d(x) + d(y_c) + d^+(x) + d^-(z) \leq 3s - 4 + 1 + 3(n - s - 1) \leq 3s - 3 + 3n - 3s - 3 \leq 3n - 6$

So the digraph is supereulerian when:

$$d(x) + d(y_c) + d^+(x) + d^-(z) \geq 3n - 5$$

From (A) and (B) and by placing $y = y_c$ we choose the larger bound, hence:

$$d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 5$$

for which D is supereulerian digraph.

∎

# 4 Conclusion

A strong digraph D of order $n \geq 4$ satisfies the following conditions: for every triple $x,y,z \in V(D)$ such that x and y are non-adjacent, if there is no arc from x to z, then $d(x)+d(y)+d^{+}(x)+d^{-}(z) \geq 3n - 5$. Then D is supereulerian.

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