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Review of Modified Gravity: Gravitational force without dark matter

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Abstract – We have derived a new modified gravitational force that is an extension of Newton's force of gravity. It shows that the observed velocity of stars inside galaxies (flat rotation curve) does not require dark matter. The new force produces logarithmic potential energy. This force is comparable to those of Gravitomagnetic force and Weber's gravitational force. It is derived from Einstein's general relativity by taking into consideration the velocity of a gravitating mass. This law treats the gravitational field as a fluid surround a gravitating mass in agreement of Maxwell assumption about the electromagnetic field. The fluid surrounding massive objects provide a background that acts like dark matter and dark energy partaking in cosmic expansion. Three modified laws of gravitation have been studied and are shown to give a velocity distribution profile in agreement with observations.

Keywords Dark matter · Flat rotation curve · Newton limit. Weber gravitational force. Gravitomagnetism

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Introduction. – The gravitational interaction connected with the motion of celestial objects has been well explained by Kepler's (Newton's) law of gravitation [1]. However, it yet fails to explain the observed velocity of stars in a galaxy [2]. The velocity distribution of stars in a galaxy is found to follow a different pattern to that anticipated by Kepler. The discrepancy between the two paradigms is known as the flatness of the rotational curve problem. It is observed that at large distances from the center of a galaxy, the velocity of a star does not decrease with distance as expected by Kepler's law. In Kepler's law, the star velocity is given by $v^2 = \frac{GM}{r}$, where r, M and G are the distance between the two masses, the central mass and

Newton's constant, respectively. Therefore, several theories are expounded to account for these discrepancies [3,4,5]. A model attempted to explain such discrepancy is proposed by Milgrom [6] in a model known as Modified Newtonian Dynamics (MOND). In the Milgrom model, a non-Keplerian Law is obtained. In addition, he introduced a concept of a universal acceleration that acts at cosmic scales amounting to a value of $10^{-10}\ m/s^2$. With this small acceleration, Milgrom was able to provide a reasonable cause of the observed flat rotation curve. An interesting model employing gravitomagnetic is recently introduced that highlighted the significance of the curvature scalar invariants [7]. Other modified gravitational models include Weber [8,9,10].

In the present paper, we present the possibility of providing a different law that yields such a velocity profile of stars in a galaxy. The new law is found to attribute gravitational interactions to additional force in addition to that one due to Newton [11]. It falls among other models like the gravitomagnetic force and the Weber gravitational law. These laws take into account the relativistic effect on the dynamics of stars. Such effect can be ignored at the scale of planets but not at star scales. We note here that these modified Newton's laws of gravitation predict a logarithmic potential Energy in addition to the Newtonian potential [12]. Of these models are the Milgrom and Webber.

Gravitomagnetic force without dark matter. – In the generalized Newton's law of Gravitation, an additional force is added to Newton's equation so that it becomes like Lorentz law of electromagnetism. Under gravity, a moving planet experiences a force that is called the gravitomagnetic force. Thus, Newton's law becomes Lorentz law of gravitation. It is found by Arbab that this law is in agrement with a flat rotation curve [11]. The gravitational force in the generalized law is expressed as

$$F=-\frac{GMm}{r^2}-\frac{mv^4}{v_c^2r} \tag{1}$$

where v_c is some reference velocity. This force is the gravitational equivalent of Lorentz's electromagnetic force. The gravitomagnetic force arises from the motion of the orbiting mass, *m* in the gravitational field of the Sun. The resulting magnetic field is analogous to that of the Biot-Savart. For ordinary velocities, Eq. (1) reduces to the ordinary Newton's law of gravitation. However, since we are interested in how stars behave at large distances, the situation will be different. For a circular motion, one has

$$\frac{mv^2}{r}=\frac{GMm}{r^2}+\frac{mv^4}{v_c^2r}\,. \tag{2}$$

Thus, the velocity can be expressed as

$$v^2=\frac{v_c^2}{2\pi}\left(1\pm\sqrt{1-\frac{4\pi GM}{v_c^2r}}\right). \tag{3}$$

Let us consider the situation where $r>\frac{4\pi GM}{v_c^2}=r_0$ and approximate the square root in the eq.(3) to obtain

(4) $$v_-^2 = \frac{GM}{r} \,,$$

and

$$v_+^2 = \frac{v_c^2}{\pi}\left(1 - \frac{\pi GM}{v_c^2 r}\right) \quad . \qquad (5)$$

Interestingly, Eq.(4) and (5) predict the overall behavior of stars in a galaxy. The ordinary Kepler's velocity relation is given by Eq.(4). For a constant velocity, Kepler's relation implies that $M \propto r$ indicating that the mass increases with distance. Such mass does not show up as luminous matter and hence it is termed dark matter. However, Eq.(5) gives a velocity distribution that is consistent with the observed flat rotation curve. To show this fact, Arbab plotted the velocity distribution in Eq.(5) that yields (Fig.1) below.

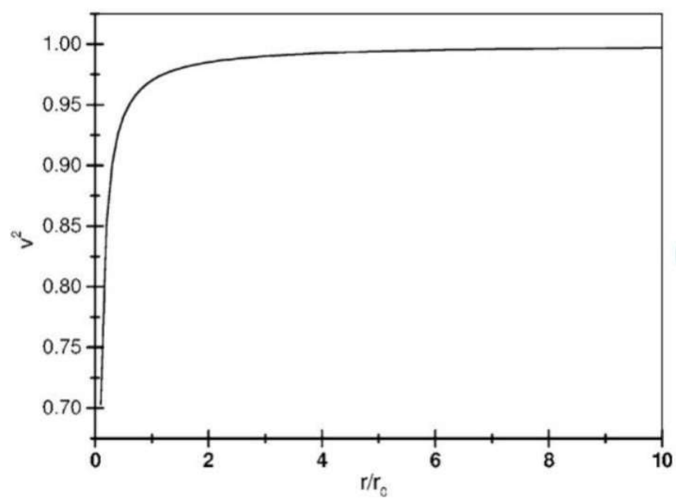

Fig.1: A flat rotation curve without dark matter [11]

The velocity here approached a maximum constant value asymptotically. One now defines the distance r_0 as the limiting distance from the central galaxy that a star can approach. It amounts to a value of a few parasec for a typical galaxy like the Milky Way. Therefore, the flattening of the velocity curve becomes predominant at distances much bigger than a few parsecs. The theoretical curve in Fig.1 corresponds to the observed curve of typical starts in elliptical galaxies. As a result, the presence of a gravitomagnetic force explains the pattern of the flat curve that is ascribed to presence of dark matter owing to Kepler. Therefore, dark matter is related to the use of Kepler law and not to a physical reality. Because of the theoretical assumption of employing Newtonian's law of gravitation, to account for all cosmic observations, dark matter exists.

A potential energy associated with the gravitomagnetic force can be derived by expressing it in terms of a distance r. To do so, we apply eq.(3) in eq.(1). This yields [11]

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$$F = -\frac{mv_c^2}{2\pi r}\left(1 \pm \sqrt{1 - \frac{4\pi GM}{v_c^2 r}}\right. \qquad (6)$$

Once again, $r >> r_0$, eq.(6) yields two forces as

$$F_{-} = -\frac{GMm}{r^2} \tag{7}$$

and

$$F_{+} = -\frac{mv_c^2}{\pi r} + \frac{GMm}{r^2} \tag{8}$$

Intriguingly, the first term in Eq. (8) is an attractive force that prevails at large distances whereas the second term is a repulsive force. Therefore, gravity is not attractive at large distances and a relativistic force acts instead. A repulsive force between masses was first proposed by Einstein in his attempt to obtain a static universe. He later declined it when he knew that the universe is expanding. That force is called the cosmological constant but it has no origin. To preserve standard form of Newton's force of gravitation, we can write the second force as

$$F_{+} = -\frac{GMm}{r^2}\left(\frac{v_c^2 r}{\pi GM} - 1\right) \Rightarrow \qquad F = -\frac{GM_{eff.}m}{r^2} \tag{9}$$

,

where

$$M_{eff} = \left(\frac{v_c^2 r}{\pi G} - M\right), \tag{10}$$

where $M_{eff.}$ is the effective mass at a distance r.

The potential energy associated with the force in eq.(8) has the form [11]

$$U_{+} = \frac{mv_c^2}{\pi}\ln\left(\frac{r}{r_s}\right) + \frac{GMm}{r} \tag{11}$$

where r_s is some reference distance from the galaxy. One can define the potential energy U_+ as an effective potential due to gravitomagnetic force. It seems that there is a gravitationally neutral matter that has a significant effect of the evolution of stars in a galaxy. This is reminiscent of the effect of a neutron (electrically neutral) on the dynamic of a nucleus. At a distance $r = r_s$, the potential energy above is reduced to a purely repulsive force.

A rival model to gravitomagnetism is the modified model of gravity (MOND) that describes the acceleration of a star by

$$a = -\frac{v_0^2}{\pi r} + \frac{GM}{r^2}, \tag{12}$$

This would generates a gravitomagnetic acceleration, at very large distances, as

$$a = -\frac{v_0^2}{\pi r.} \tag{13}$$

By comparing Eq.(13) to the acceleration of a mass with a typical acceleration (a_0), one has [1]

$$a = \frac{\sqrt{GMa_0}}{r}, \tag{14}$$

where $a_0 = 1.2 \times 10^{-10} m/s^2$. This yields the characteristic acceleration

$$a_0 \approx -\frac{v_c^4}{\pi^2 GM}. \tag{15}$$

Incidently, the MOND acceleration turns out to be the same as the gravitomagnetic acceleration. Consequently, both MOND and gravitomagnetism are capable of explaining the rotation pattern of stars in a galaxy.

Weber's gravitational force. – The Weber's gravitational force is considered as an alternative model for the Newton's law of gravitation [8,9,10]. In addition, Tisserand proposed a new model of gravitation that depends on the object velocity to resolve the mercury anomaly problem. This additional correction of this model to Newton's one involves a factor of $\frac{1}{c^2}$ compare to that in [8]. The two models yield a flat rotation curve in agreement with that of gravitomagnetic model of gravity. They thus resolve the dark matter and dark energy problems. We will start here by studying the Weber's gravitational force (WGF) [8]. It is given by

$$F = -\frac{GMm}{r^2}\left(1 - \frac{\dot{r}^2}{c^2} + \frac{r\ddot{r}}{c^2}\right), \tag{16}$$

where M and m are the central and the orbiting masses. The terms $\dot{r}^2$ and $\ddot{r}$ represent the rate of change of distance and velocity between the two masses. The first term is that of Newton's law of gravitation, which varies inversely with

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the square of the distance for ordinary velocities. The second term is related to cosmological phenomena. Moreover, the motion composes of receding and approaching relative accelerations. We are interested here in studying the behavior of stars at large distances, where the star moves at a relatively high speed compared to planets. Let us describe the case of circular motion

$$\frac{mv^2}{r} = \frac{GMm}{r^2} - \frac{GMmu^2}{c^2 r^2}, \qquad u = \dot{r}. \tag{17}$$

Now, we will solve Eq.(17) as a function of distance r and velocity v. It gives

$$v^2 = \frac{GM}{r} - \frac{GMu^2}{c^2 r}. \tag{18}$$

And by considering the case when $r > \frac{GM}{c^2} = r_0$, Eq.(18) yields

$$v_-^2 = \frac{GM}{r} \tag{19}$$

$$v_+^2 = -\frac{GMu^2}{c^2 r} \tag{20}$$

Eq.(19) and (20) represent the full behavior of a star orbiting a galaxy. The ordinary Kepler velocity equation is

Eq.(19). The velocity a distribution in Eq.(20) agrees with the gravitomagnetic finding that yields results conform with the observed flat rotation curve. Plotting the velocity distribution in Eq.(20) shows the pattern as (Fig.2) below

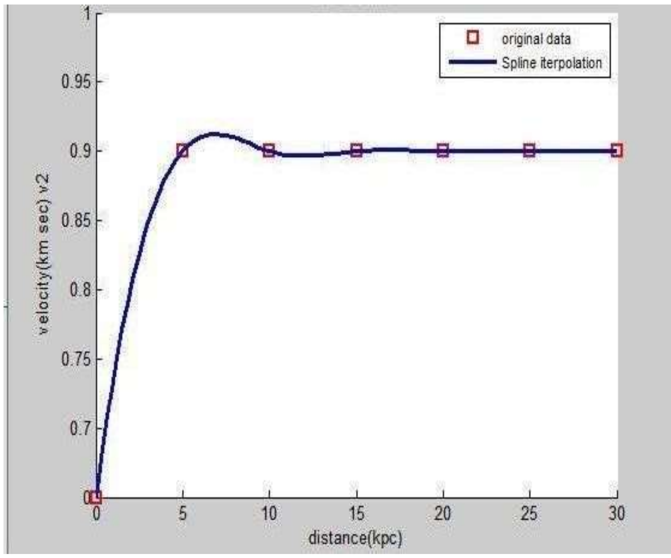

Fig.2: Velocity profile of observed stars in a galaxy.

The theoretical curve agrees with observational findings of gravitomagnetic and with the observation of stars motion in galaxies. Therefore, the Weber gravitational force (WGF) also resolves the dark matter problem manifested in using Kepler law. Now the third term in Eq.(16) becomes

$$F = -\frac{GMma}{r\ c^2} \tag{21}$$

where *a* is the cosmological acceleration constant. There is an indication that *a* is currently negative (a negative sign represents a deceleration) [9]. Therefore, the force in Eq.(21) in this situation is repulsive and is proportional to $\frac{1}{r}$. However, if gravity is repulsive, it is important to assume that Newton's law of gravitation will apply up to some distance. Beyond this point, cosmic acceleration will continue unabated. Then, the dark energy would affect the gravitational interaction between orbiting masses. The dark dark energy allows the space in which stars exist to expand.

Modified Newton Limit (MONL) without dark matter. – We would like here to derive a gravitational law by considering the gravitating mass star velocity to be important and can't be neglected. In doing so, we will get a velocity-dependent gravitational force. It reduces to the Newton's equation when the velocity is considered low. Such an effect can be derived from the geodesic equation by keeping velocity terms. These terms are usually ignored when Newton's law is assumed to be a limiting gravitational law to Einstein's general relativity [1]. We trust such effect can modify

Newton's law to rule relativistic masses lying at large scales that will reshape Newton as well as Einstein's equations. The path of a particle (mass) in general space-time is described by the geodesic equation as [1]

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \qquad (\mu,\nu,\lambda = 0,1,2,3) \tag{22}$$

where $x^\mu = (ct, \vec{r})$ defines the particle position and τ is the proper time measured by an observer at rest with respect to a clock, and $\Gamma^\mu_{\nu\lambda}$ is known as Christoffel symbols. The Chrsitoffel symbols are defined by [1]

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\alpha}\left(\partial_\nu g_{\alpha\lambda} + \partial_\lambda g_{\alpha j} - \partial_\alpha\, g_{\nu\lambda}\right). \tag{23}$$

However, at low velocity, one has $\frac{dx^\mu}{d\tau} \approx \frac{dx^i}{dt}$, with i = (1,2,3) denoting the spatial dimensions.

In order to get Newton's gravity equation with velocity-dependent force, we don't neglect velocity terms in expanding eq.(22). The expansion of the Christoffel symbols terms yields

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0, \tag{24}$$

where *i,j,k* are the spatial components. We then calculate Γ^i_{00} and $\Gamma_{jk}{}^i$ and Γ^i_{00} assuming non-negligible velocities limit. We approximate the gravitational field, the metric tensor, to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a function of velocity and coordinate (r). Substituting this in Eq.(23) yields, for spatial components,

$$\Gamma^i_{jk} = -\frac{1}{2}\partial_i\, h_{jk}, \tag{25}$$

where the particle velocity vector, $u^\mu = (\gamma c, \gamma\vec{u})$. The acceleration now becomes

$$\frac{d\vec{u}}{dt} = -\vec{\nabla}\phi + \vec{\nabla}\left(\frac{h_{11}u_x^2}{2}\right) - \frac{h_{11}}{2}\vec{\nabla}u_x^2 + \nabla\left(\frac{h_{22}u_y^2}{2}\right) - \frac{h_{22}}{2}\vec{\nabla}u_y^2 + \vec{\nabla}\left(\frac{h_{33}u_z^2}{2}\right) - \frac{h_{33}}{2}\vec{\nabla}u_z^2, \tag{26}$$

where we have used the vector identity, $\left(\vec{\nabla}f\right)g = \vec{\nabla}(fg) - f\left(\vec{\nabla}g\right)$, in the above equation.

Let us now assume, $u_x = u_y = u_z = u$ and $h_{11} = h_{22} = h_{33} = h$, then eq.(26) becomes

$$\frac{d\vec{u}}{dt} = -\vec{\nabla}\phi + \vec{\nabla}\left(\frac{hu^2}{2}\right) - \frac{h}{2}\vec{\nabla}u^2, \tag{27}$$

or

$$\frac{d\vec{u}}{dt} = -\vec{\nabla}\phi + \vec{\nabla}\left(\frac{hu^2}{2}\right) - h(\vec{\nabla}u).\,\vec{u} \tag{28}$$

Assuming the velocity does not change appreciably with distance, one can neglect the last term on the right hand-side in eq.(28). The resulting equation now becomes

$$\frac{d\vec{u}}{dt} = -\vec{\nabla}\left(\phi + \frac{hu^2}{2}\right) + h(\vec{\nabla}u^2) \tag{29}$$

The Nervier-Stokes equation of an incompressible fluid (*e.g.*, Euler's equation for an inviscid flow) is given by [6]

$$\rho\frac{Du}{Dt} = \rho g - \nabla P + \mu\nabla^2 u\,, \tag{30}$$

where ρ is the fluid density and $\rho\frac{Dv}{Dt}$ is the force density acting on the fluid. The second and third terms on the right hand-side in eq.(30) are the pressure and viscosity force densities. Intriguingly, the modified gravitational equation in eq.(29) is analogous to the Navier-Stokes equation of a viscous fluid [13]. This urges us to look at the gravitational field around massive objects as a viscous fluid. The notion of a field to behave like a fluid was noted by Maxwell for the electromagnetic field. The gravitational force density is defined by $g\rho = -\vec{\nabla}(\rho\phi)$ and the pressure force is defined by $P = \frac{1}{2}\,\rho\, u^2$ and the viscosity $\mu = \rho h$. This furnishes the analogy between the two paradigms.

Now if we assume u to be a slowly varying function with distance, eq.(29) can be approximated to

$$m\frac{d\vec{u}}{dt} \approx -m\nabla\phi + \frac{mu^2}{2}\,\vec{\nabla}h\,, \tag{31}$$

where $\phi = -\frac{GM}{r}$ is the gravitational potential. If we want the second term in eq.(31) to behave like a gravitomagnetic force, then $h = -2A\ln r$ so that

$$g_{00} = -\vec{\nabla}\phi - \vec{\nabla}(Au^2 \ln r). \tag{32}$$

This leads to a velocity-dependent gravitational force. of the form

$$F = -\frac{GMm}{r^2} - \frac{mAu^2}{r} \tag{33}$$

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which reduces to Newton's law of gravitation for very small velocities. For all practical uses velocity u does not vary (constant) much with distance so that we can call it $u = v_s$. It resembles the velocity v_c in the gravitomagnetic model. For a rotating mass, eq.(33) can be expressible in the form

$$\frac{mv^2}{r} = \frac{GMm}{r^2} + \frac{mA{v_s}^2}{r} \tag{34}$$

The above equation reveals that the velocity $v = v_s$ at large distance from the center of gravitating mass. In such a case the first term would be insignificant. This is exactly what is observed for stars inside a galaxy. However, Kepler's law could explain this behavior if dark matter is imposed. Therefore, the derived velocity-dependent law in eq.(33) does indeed resolve the flat rotation curve without a need of dark matter. It is at the footing as that of gravitomagnetic and MOND models.

One can now place an estimate to the constant A in eq.(33) by referring to the universal acceleration a_0. In this case, one finds

$$A = \frac{a_0 r}{v_s^2} \approx 1\,, \tag{35}$$

at a cosmic level with $r \sim 10^{26} m$, $v_s \sim c$.

Concluding remark

We have considered in this paper the effect of higher velocity terms in the geodesic equation, which are usually neglected when Einstein's equation reduces to Newton's equation. These terms are important when considering the motion of stars in a galaxy. The resulting terms modifies Newton's equation by a velocity-dependent term that is able to account for the observed flat rotation curve shown by stars in a galaxy. The new terms give rise to new potential (logarithmic). The new velocity-dependent gravity law treats the gravitational field as a fluid filling the space around a central mass. It mimics the picture Maxwell endowed to the electromagnetic field a long time ago. We compare the Weber, gravitomagnetic and MOND laws of gravitation and show how they could explain the flat rotation curve observed by starts in a galaxy. All of the four models are capable to account convincingly well to the velocity profile exhibited by stars in a galaxy. Besides the dark matter, our model could also help explain the origin of dark energy causing space to expand and accelerates. The presented models predict a space acceleration $a_0 \sim 10^{-10} m/s^2$ driving the universe expansion at the present time.

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