

Review of Gamow-Teller and Fermi transition strength functions

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Abstract

We studied the temperature effect in isospin-singlet pairings in Gamow-Teller excitations. We use theories of a hole-particle in the mean-field shell model to study decay transition using the one-particle-one-hole model for the β -decay of odd-even isotopes and the two-particle-hole models for the β -decay of even-even and/or odd-odd isotopes. Our reference isotopes for the one-particle-one-hole model are ^{15}O , ^{15}N , ^{17}F , and ^{41}Sc , whereas for the two-particle-hole model we use ^{16}N (for β^- -decay) and ^{56}Ni and ^{40}Sc (for β^+ /EC).

The calculations involve evaluating the matrix elements of Gamow-Teller and Fermi transitions, then calculating the reduced transition probabilities of Gamow-Teller and Fermi, from which we evaluate the half-lives and the strength function ft . The results are compared with the available experimental data. For the one-particle-one-hole model, we found there is a deviation from experimental values which indicates that the model is not valid for beta decay for the even-even nuclei in the ground state due to the residual nucleon-nucleon interaction. As for a two-particle-hole model, we calculated the transition amplitude, from which we calculated the strength of the transition $\log ft$ values. We found an excellent agreement between experimental and theoretical results.

By drawing the relationship between temperature versus $\log ft$ values, we found the general trend is that the strength function values slowly decrease as temperatures increase. There are fluctuations $\log ft$ due to the strongly dependent of $\log ft$ on the shell configuration of the valence nucleons.

Keywords

Nuclear physics, Gamow-Teller transition, Fermi transition, β -decay.

1. Introduction

Nuclear β -decay plays an important role in nuclear physics in particular and in various branches of science in general, such as astrophysics and particle physics. The investigation of β -decay provides valuable insights into how effective nuclear interactions depend on the spin and isospin, as well as on nuclear properties such as masses, shapes, and level densities [1, 2]. In astrophysics, β -decay is responsible for the formation of neutron stars, the factories of heavy elements in our universe [3] by setting the time scale of the rapid neutron-capture via the half-lives of β -decay. In particle physics, β -decay offers the first experimental evidence of parity violation [4], and is utilized to verify the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [5].

Beta decay results from the presence of a weak force, which undergoes a relatively slow decay time. Nucleons are formed from up and down quarks, and the quark has a weak strength which contributes

to changing the flavor of lepton by producing the W boson, which produces an electron/antineutrino or positron/neutrino pair. The most obvious example is the decay of a neutron, which consists of an up quark and two down quarks, to produce a proton that consists of two up quarks and a down quark.

The process of measuring the beta decay strength functions, and hence the decay half-lives, in an accurate way, is a required and necessary method and can be applied Beyond the Standard Model (BSM) as a way to search for new fundamental physics can be discovered through beta decay in atomic nuclei. The problem of understanding the 'background' of the Standard Model and its related perception of the low-energy quantum chromodynamic effects that appear in the form of a nuclear structure is one of the outstanding difficulties hampering the discovery of new physics.

The difficulty of treating experimentally-favored nuclei theoretically in a framework that allows for measured uncertainty makes the issue worse. Nevertheless, advancements have been achieved over the past several decades that allow for the systematic construction of the internucleon interaction within a strong field theory framework. The medium-mass nuclei, frequently relevant for BSM searches, can now be treated ab initio, thanks to developments in the many-body theory and computer power [6]. Of course, there is still much to be learned about effective-field theory and how approximation strategies used in ab initio computations affect the relevant observables.

With the achieved advances in the measurement of nuclear β -decay half-lives due to the development of radioactive ion-beam facilities, a complete listing of the experimental data is available nowadays [7]. It is imperative to test the fundamental theoretical model and their abilities to reproduce the experimental values to determine the weak and/or strength points in these models.

This review study presents the latest conventions in the theory of β -decay theory and procedures to evaluate the transition matrices due to the β -decay [8]. Then focuses on β -decay transition in one-particle-hole and two-particle-hole nuclei schemes. The one-particle-hole theory is utilized to predict the half-lives of odd-even nuclei: ^{15}O , ^{17}F , ^{39}Ca , and ^{41}Sc . whereas, the two-particle-hole theory is utilized to calculate the strength functions of the EC/β^+ -decay of ^{56}Ni as an even-even nucleus, the β -decay of ^{16}N as an odd-odd nucleus, and finally the EC/β^+ -decay of ^{40}Sc as an odd-odd nucleus. The reason for choosing these isotopes is to study the effect of residual NN force on the valence nucleons. It is expected that such a force plays a greater role for an even number of nucleons than an odd number [9] which suppresses the single particle transitions. The temperature effect on the strength functions is presented. Finally, the study presents conclusions and suggestions for further studies.

2. Theoretical background

2.1. Theory of nuclear β -decay

In the nuclear scale the β^- -decay is written as

$${}^A_Z X_N \rightarrow {}^A_{Z+1} Y_{N-1} + e^- + \bar{\nu}_e. \quad (1)$$

The nuclear β^+ -decay is

$${}^A_Z X_N \rightarrow {}^A_{Z-1} Y_{N+1} + e^+ + \nu_e. \quad (2)$$

Finally nuclear EC reads

$${}^A_Z X_N + e^- \rightarrow {}^A_{Z-1} Y_{N+1} + \bar{\nu}_e. \quad (3)$$

In the three processes, shown in fig.(2), the parent nucleus ${}^A X$ and the daughter nucleus ${}^A Y$ are isobars, i.e. both have the same mass number A . This process has a coupling constant G_F which is not fundamental. It involves two fundamental vertices of weak coupling g_W . The strength of weak interaction is measured in muon decay, shown in fig.(1), where $q^2 < m_\mu c^2 = 106 \text{ MeV}$. Thus the W -boson propagator in the natural unit can be written

$$\frac{-i (g_{\mu\nu} - q_\mu q_\nu / m_W^2)}{q^2 - m_W^2} \approx \frac{i g_{\mu\nu}}{m_W^2}.$$

In muon decay this becomes g_W^2/m_W^2 . Hence, the weak coupling g_W can be related to G_F using [10]

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8(m_W c^2)^2}. \quad (4)$$

This is valid for large mass of W -boson and small energy q^2 of β -decay, i.e. $q^2 \ll (m_W c^2)^2$. In case of $q^2 \geq (m_W c^2)^2$ the weak interaction is more probable than electromagnetic force. In other words, the weak interaction is only weak because of the large W -boson mass ($m_W = 80.403 \pm 0.029 \text{ GeV}/c^2$). For muon decay $G_F = 1.16639(1) \times 10^{-5} \text{ GeV}^{-2}$ [10].

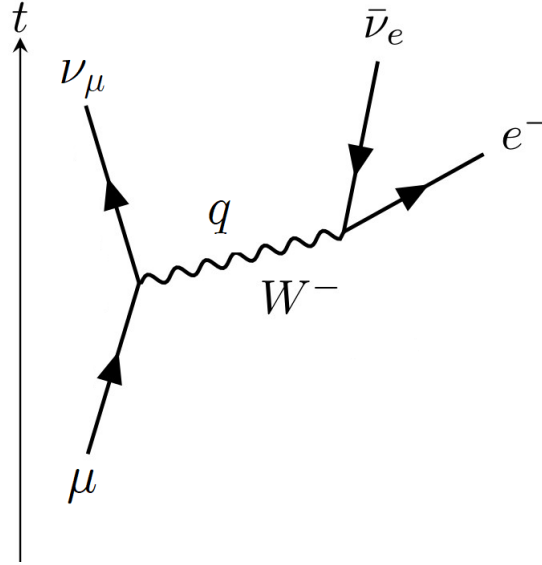


Figure 1: Feynman diagram depicts the weak muon decay. The W -boson propagator carries momentum q , where $q^2 \ll (m_W c^2)^2$, for precise measurements of the weak coupling g_W .

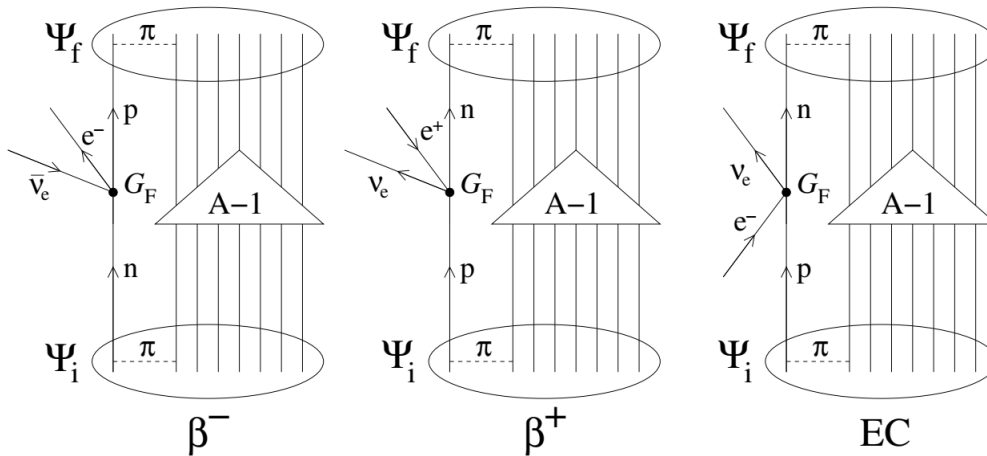


Figure 2: Nuclear β^- , β^+ , and EC decay diagrams. Only one nucleon contributes to the β decay process whereas the remaining $A - 1$ nucleons are spectators. The initial and final states Ψ_i and Ψ_f are the initial and final nuclear states of a strongly interacting A -body wave function. G_F is the pointlike strength of the weak interaction vertex.

2.2. Allowed β decay

We consider the lepton final state emitted in s -state ($l = 0$) relative to the nucleus (isotropic emission). Similarly, in allowed EC the initial electron is from an s -shell, and the final neutrino is in an s state relative

to the nucleus. Other β -decay processes involving higher values of leptons orbital angular momentum (p -state, d -state, etc.) are traditionally called forbidden beta transitions ¹.

The general β -decay $N_1 \rightarrow N_2 + \text{lepton} + \text{antilepton}$ means each lepton carries spin $s = \frac{1}{2}$. The β^\pm decay final leptonic state can couple to total spin $s_{\text{leptons}} = 0, 1$. In the EC process the initial proton and electron can couple to $j \pm \frac{1}{2}$ and the final neutrino can couple to $j \pm \frac{1}{2}$ or $j \mp \frac{1}{2}$. Thus in all cases, the lepton spin can change nuclear total angular momentum J by 0 or 1. In allowed β -decay, transitions with no angular momentum change are called Fermi transitions and those with total angular momentum change by one unit are called Gamow-Teller transitions [11]. There is no source for parity changing allowed β -decay transition $(-1)^{\Delta I} = +1$ [11]. In the standard model, the lepton number is considered to be conserved separately for each lepton flavor: electron e , muon μ , and tau τ .

Table 1: Electric charge q , baryon number B , lepton number L , and mass m for fermions involved in the β -decay.

particle	q	B	L	m (MeV/ c^2)
electron (e^-)	$-e$	0	+1	0.511
positron (e^+)	$+e$	0	-1	0.511
electron neutrino (ν_e)	0	0	+1	0
electron antineutrino ($\bar{\nu}_e$)	0	0	-1	0
proton (p)	$+e$	+1	0	938.3
neutron (n)	0	+1	0	939.6

Table 2: Selection rule for allowed β -decay transitions. Here $J_i(J_f)$ is the angular momentum of the initial (final) nuclear state and correspondingly for the parity π [12].

Type of transition	$\Delta J = J_f - J_i $	$\pi_i \pi_f$
Fermi	0	+1
Gamow-Teller	$1(J_i = 0 \text{ or } J_f = 0)$	+1
Gamow-Teller	$0, 1(J_i > 0, J_f > 0)$	+1

2.3. Half-lives, reduced transition probabilities, and ft values

Half-life represented by $t_{\frac{1}{2}}$ is computed from transition probability \mathcal{T}_{fi} ,

$$t_{\frac{1}{2}} = \frac{\ln 2}{\mathcal{T}_{fi}}, \quad (5)$$

\mathcal{T}_{fi} is calculated Fermi golden rule of time-dependent perturbation theory [13] to get

$$t_{1/2} = \frac{\kappa}{f_0(B_F + B_{GT})}, \quad (6)$$

where κ (kappa) is a constant [12]

$$\kappa = \frac{2\pi^3 \hbar^7 \ln 2}{m_e^5 c^4 G_F^2} = 6147s, \quad (7)$$

¹Forbidden does not indicate that the transition is completely not allowed. The vast contribution to β -decay transition is due to s -state leptonic emission.

f_0 is the lepton kinematics phase space integral, B_F and B_{GT} are the Fermi and Gamow-Teller reduced transition probabilities needed to be calculated, respectively. They can be broken up into factors [8],

$$B_F = \frac{g_V^2}{2J_i + 1} |\mathcal{M}_F|^2, \quad (8)$$

and

$$B_{GT} = \frac{g_A^2}{2J_i + 1} |\mathcal{M}_{GT}|^2, \quad (9)$$

where J_i is the total angular momentum (nuclear spin) of the initial nuclear state. g_V and g_A are coupling constants for vector current and axial current, respectively [14]. \mathcal{M}_F and \mathcal{M}_{GT} are the interaction amplitudes. The quantity $f_0 t_{\frac{1}{2}}$ (written as ft value) represents the allowed β -decay transition. It depends on the nuclear structure, which is contained in the reduced matrix elements. In ref.[13] it has been called the reduced half-life or comparative half-life. The vector coupling constant $g_V = 1.0$, its value is determined by conserved current $j^\mu = \frac{1}{2} \bar{\psi} \gamma^\mu \psi$ [8].

The factor $g_A = 1.25$, is the axial vector coupling constant of the weak interaction determined by partially conserved axial vector current $j_A^\mu = \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^5 \psi$. All those currents are calculated using the standard model. g_A is affected by many-nucleon correlation (pairing residual interaction) value reduced by 20-30% is some times used [8]. The presence of both vector and axial vector coupling constant in $t_{\frac{1}{2}}$ relation (6) reflects the parity non-conserving nature of the weak interaction [4]. Vectors have parity properties, $\vec{V}(-\vec{r}) = -\vec{V}(\vec{r})$ under space inversion. On the other hand, axial vector (pseudo vector) \vec{A} are invariant under space inversion,

$$\vec{A}(-\vec{r}) = +\vec{A}(\vec{r}).$$

For lepton current the violation of parity conservation is maximal, and the weak interaction amplitude for the leptonic contribution contains the combination $V - A$ in equal division. This holds in the quark level of the hadrons [10]. The hadronic current

$$j \propto V - \left(\frac{g_A}{g_V}\right)A = V - (1.25A). \quad (10)$$

Thus the $V - A$ current is proportional to

$$V - A \propto \bar{\psi} \gamma^\mu (1 - \gamma^5) \psi.$$

The minus sign is an indication of the left-handedness of the Leptons involved in weak interactions. Since the ft value is very large can be suppressed by logarithm,

$$\log ft = \log_{10}(f_0 t_{\frac{1}{2}} [s]). \quad (11)$$

2.4. Wigner-Eckart theorem

Assume $\mathcal{T}_q^{(k)}$ is spherical tensor operator (such as angular momentum operators) acts on angular momentum basis $|jm\rangle$. Transition amplitude due to the tensor operator is given by [15, 16]

$$\langle \xi_f; j_f m_f | \mathcal{T}_q^{(k)} | \xi_i; j_i m_i \rangle = \mathcal{M} \delta_{m_f, m_i + q}$$

$\mathcal{M} = 0$ unless $m_f = m_i + q$. This is the Wigner-Eckart Theorem. The matrix elements of tensor operators with respect to angular-momentum eigenstates satisfy [16]

$$\langle \xi'; j' m' | \mathcal{T}_q^{(k)} | \xi; j m \rangle = \langle j m; k q | j k; j' m' \rangle \frac{\langle \xi' j' || \mathcal{T}^{(k)} || \xi j \rangle}{\sqrt{2j + 1}}, \quad (12)$$

where the double-bar matrix element is independent of m and m' , and q . Here ξ this amplitude represents transition from $|\xi; jm\rangle$ to $|\xi'; j'm'\rangle$. Before we present a proof of this theorem, let us look at its significance. First, we see that the matrix element is written as the product of two factors. The first factor is a Clebsch-Gordan coefficient for adding j and k to get j' . It depends only on the geometry—that is, on the way the system is oriented with respect to the z -axis. There is no reference whatsoever to the particular nature of the tensor operator. The second factor does depend on the dynamics; for instance, ξ may stand for the radial quantum number, and its evaluation may involve, for example, the evaluation of radial integrals. On the other hand, it is completely independent of the magnetic quantum numbers m , m' , and q , which specify the orientation of the physical system. To evaluate $\langle \xi', j'm' | \mathcal{T}_q^{(k)} | \xi, jm \rangle$ with various combination of m , m' , and q' it is sufficient to know just one of them: all others can be related geometrically because they are proportional to Clebsch-Gordan coefficients, which are known. The common proportionality factor is $\langle \xi', j' \| \mathcal{T}^{(k)} \| \xi j \rangle$, which makes no reference whatsoever to the geometric features. The selection rules for the tensor operator matrix element can be immediately read off from the selection rules for adding angular momentum. Indeed, from the requirement that the Clebsch-Gordan coefficient be nonvanishing, we immediately obtain the m -selection rule derived before and also the triangular relation $|j - k| \leq j' \leq j + k$.

There are different conventions for the reduced matrix elements. One convention that includes an additional phase and normalization factor with the aid of $6j$ symbol [15, 17]

$$\langle \xi'; j'm' | \mathcal{T}_q^{(k)} | \xi; jm \rangle = (-1)^{j-m} \begin{Bmatrix} j' & k & j \\ -m' & q & m \end{Bmatrix} \langle \xi' j' \| \mathcal{T}^{(k)} \| \xi j \rangle. \quad (13)$$

2.5. Fermi and Gamow-Teller matrix element

In the beginning, let us review the scales of β -decay we need to evaluate the transition matrices for. They are as follow

1. Quark scale. According to standard model the β^- -decay is attributed to weak flavor symmetry of u and d quarks, according to

$$u \rightarrow d + e + \bar{\nu}_e.$$

2. Nucleon scale. The β^- -decay is due to decay of free (or quasi-free) neutron, according to

$$n \rightarrow p + e + \nu_e.$$

3. Nuclear scale, where the β^- -decay is due to the nuclear decay given in eq.(1)

For nucleon scale β^- -decay, we denote the proton using index a or f , and the neutron using index b or i . Whereas for β^+ -decay, we denote the neutron using index a or f , and the proton using index b or i .

Fermi matrix element \mathcal{M}_F [18] and Gamow-Teller (GT) matrix element \mathcal{M}_{GT} [19] are the most important values needed to be calculated using the initial and final nuclear wave functions which carry the nuclear structure information. Fermi operator is just the unit operator $\hat{\mathbf{1}}$. GT operator is the Pauli spin operator $\hat{\boldsymbol{\sigma}}$. These operators are the simplest scalar and axial vector operators that can be constructed. The selection rules are shown in table (2).

The Fermi and Gamow-Teller matrix can be written as [8]

$$\mathcal{M}_F = \langle \xi_f J_f \| \hat{\mathbf{1}} \| \xi_i J_i \rangle = \delta_{J_i J_f} \sum_{a,b} \mathcal{M}_F(fi) \langle \xi_f J_f \| [c_f^\dagger \tilde{c}_i]_{\Delta J=0} \| \xi_i J_i \rangle, \quad (14)$$

and

$$\mathcal{M}_{GT} = \langle \xi_f J_f \| \hat{\boldsymbol{\sigma}} \| \xi_i J_i \rangle = \sum_{a,b} \mathcal{M}_{GT}(fi) \langle \xi_f J_f \| [c_f^\dagger \tilde{c}_i]_{\Delta J=1} \| \xi_i J_i \rangle, \quad (15)$$

where $\mathcal{M}_F(fi)$ and $\mathcal{M}_{GT}(fi)$ are the single-particle matrix for Fermi and GT, respectively. They can be written as [15, 17]

$$\begin{aligned}\mathcal{M}_F(ab) &= \mathcal{M}_F(fi) = \langle f \parallel \hat{\mathbf{1}} \parallel i \rangle = \delta_{fi} \hat{j}_f \\ &= \langle n_f l_f j_f \parallel \hat{\mathbf{1}} \parallel n_i l_i j_i \rangle = \delta_{n_f n_i} \delta_{l_f l_i} \delta_{j_f j_i} \hat{j}_f,\end{aligned}\quad (16)$$

and [15, 17]

$$\begin{aligned}\mathcal{M}_{GT}(ab) &= \mathcal{M}_{GT}(fi) = \frac{1}{\sqrt{3}} (f \parallel \hat{\boldsymbol{\sigma}} \parallel i) = \frac{1}{\sqrt{3}} \langle n_f l_f j_f \parallel \hat{\boldsymbol{\sigma}} \parallel n_i l_i j_i \rangle \\ &= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \times 2 \delta_{n_f n_i} \delta_{l_f l_i} \hat{j}_f \hat{j}_i (-1)^{l_f + j_f + \frac{3}{2}} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ j_f & j_i & l_f \end{array} \right\}, \\ &= \sqrt{2} \delta_{n_f n_i} \delta_{l_f l_i} \hat{j}_f \hat{j}_i (-1)^{l_f + j_f + \frac{3}{2}} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ j_f & j_i & l_f \end{array} \right\}.\end{aligned}\quad (17)$$

2.6. Symmetry properties of SP-matrix element

Since we do not have orbital degrees of freedom, the symmetry properties for Fermi single-particle matrix is

$$\mathcal{M}_F(ab) = \mathcal{M}_F(ba), \quad (18)$$

which means that Fermi transitions for β^+ and β^- are similar. For the GT transition, we have

$$\mathcal{M}_{GT}(ab) = (-1)^{j_a + j_b + 1} \mathcal{M}(ba). \quad (19)$$

The Gamow-Teller single-particle matrix element for the lowest l_j combinations are independent of n as long as $\Delta n = 0$ and thus they obey the selection rule $\Delta l = 0$. Table (3) gives the GT SP-matrix. These matrix elements are calculated using a C++ function based on eq.(17).

Table 3: Gamow-Teller single-particle matrix elements.

a/b	$s_{\frac{1}{2}}$	$p_{\frac{3}{2}}$	$p_{\frac{1}{2}}$	$d_{\frac{5}{2}}$	$d_{\frac{3}{2}}$	$f_{\frac{7}{2}}$	$f_{\frac{5}{2}}$	$g_{\frac{9}{2}}$
$s_{\frac{1}{2}}$	$\sqrt{2}$	0	0	0	0	0	0	0
$p_{\frac{3}{2}}$	0	$2\frac{\sqrt{5}}{3}$	$-\frac{3}{4}$	0	0	0	0	0
$p_{\frac{1}{2}}$	0	$\frac{4}{3}$	$-\frac{\sqrt{2}}{3}$	0	0	0	0	0
$d_{\frac{5}{2}}$	0	0	0	$\sqrt{\frac{14}{5}}$	$-\frac{4}{\sqrt{5}}$	0	0	0
$d_{\frac{3}{2}}$	0	0	0	$\frac{4}{\sqrt{5}}$	$-\frac{2}{\sqrt{5}}$	0	0	0
$f_{\frac{7}{2}}$	0	0	0	0	0	$2\sqrt{\frac{6}{7}}$	$-4\sqrt{\frac{2}{7}}$	0
$f_{\frac{5}{2}}$	0	0	0	0	0	$4\sqrt{\frac{2}{7}}$	$-\sqrt{\frac{19}{7}}$	0
$g_{\frac{9}{2}}$	0	0	0	0	0	0	0	$\frac{1}{3}\sqrt{\frac{110}{3}}$

2.7. Phase-space factors

The half-life contains the integrated leptonic phase space which is called a phase-space factor f_0 . Some references call it Fermi integral. For β^\pm -decay, the phase-space factor is [7]

$$f_0^{(\pm)} = \int_0^{E_0} F_0(\mp Z_f, \varepsilon) p \varepsilon (E_0 - \varepsilon)^2 d\varepsilon, \quad (20)$$

F_0 is the Fermi function. \mathcal{E} is the energy ratio given by

$$\mathcal{E} = \frac{E_e}{m_e c^2}, \quad (21)$$

where E_e is the total energy of the emitted electron or positron. E_0 denotes the nuclear energy difference

$$E_0 = \frac{E_i - E_f}{m_e c^2}, \quad (22)$$

where E_i and E_f are the initial and final energies, respectively, for nuclear states. The momentum is given by

$$p = \sqrt{\mathcal{E}^2 - 1}. \quad (23)$$

For electron capture the phase-space factor is [7]

$$f_0^{(EC)} = 2\pi(\alpha Z_i)^3(\mathcal{E}_0 + E_0)^2, \quad (24)$$

where

$$\mathcal{E}_0 = \frac{m_e c^2 - \mathcal{B}}{m_e c^2} \approx 1 - \frac{1}{2}(\alpha Z_i)^2, \quad (25)$$

where \mathcal{B} is the atomic binding energy of the captured electron usually (1s orbital) and

$$\alpha = \frac{e^2/4\pi\epsilon_0}{\hbar c} = \frac{1}{137}, \quad (26)$$

is the fine structure constant. The approximation for \mathcal{E}_0 in eq.(25) is valid for $\alpha Z_i \ll 1$, which holds for light nuclei $Z_i < 40$. The phase-space factors eq.(20) or eq.(24) are functions of the nuclear energy difference E_0 . The final state for β^\pm -decay involves a three-body state (one baryon and two leptons), which reflects a complicated kinematics in E_0 dependence of $f_0^{(\pm)}$. In EC the final state is a two-body state and the energy-momentum conservation result in a definite energy for the emitted neutrino. The β^\pm -decay spectrum of the emitted electron or positron is continuous due the distribution of energy among the three body systems. An example of the spectrum is shown in fig.(3).

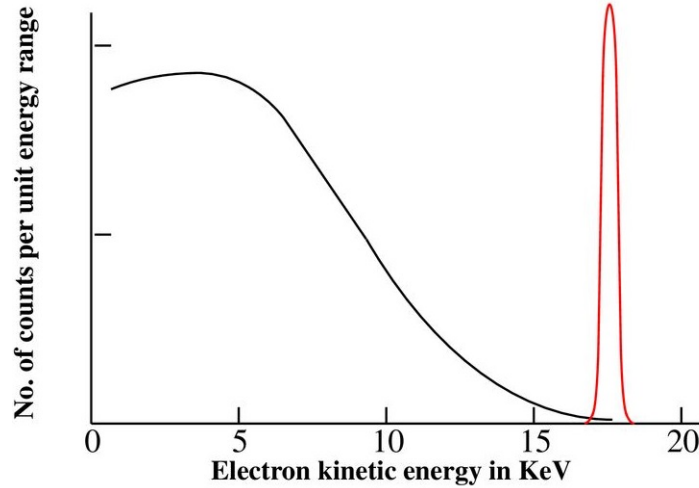


Figure 3: The β decay spectrum of tritium (${}^3\text{H} \rightarrow {}^3\text{He}$). The red sharp spectra are for neutrinoless decay. Taken from [20].

The Fermi function (20) can be approximated using non-relativistic analytical technique, known as *Primakoff-Rosen approximation* [21]

$$F_0(Z_f, \mathcal{E}) \approx \frac{\mathcal{E}}{p} F_0^{(PR)}(Z_f); \quad F_0^{(PR)(\pm)}(\mp Z_f) = \frac{2\pi\alpha Z_f}{1 - e^{\mp 2\pi\alpha Z_f}}. \quad (27)$$

This approximation yields good results for high Q values. We can expand the phase-space factor in eq.(20) [7, 21]

$$f_0^{(\pm)} \approx \frac{1}{30} (E_0^5 - 10E_0^2 + 15E_0 - 6) F_0^{(PR)(\pm)} (\mp Z_f). \quad (28)$$

2.8. β -decay Q -values

The Q -value for any nuclear reaction or decay is given by

$$Q = K_f + K_i = E_i - E_f. \quad (29)$$

Using eq.(22), For β^- -decay we have

$$E_0 = \frac{Q_{\beta^-} + m_e c^2}{m_e c^2}. \quad (30)$$

For β^+ -decay we have

$$E_0 = \frac{Q_{\beta^+} + m_e c^2}{m_e c^2}. \quad (31)$$

Finally, for EC

$$E_0 = \frac{Q_{EC} - m_e c^2}{m_e c^2}. \quad (32)$$

The Q -values for all decays are listed in [7]. E_0 represents the endpoint energy of the decay. The decay half-life can be calculated directly once the one-body transition densities

$$\langle \xi_f J_f \parallel [c_a^\dagger \tilde{c}_b] \parallel \xi_i J_i \rangle, \quad (33)$$

are known.

2.9. Classification of β -decay

The classifications of β -decay is done interms of $\log ft$ values, given in table (4) [22].

2.9.1 Superalowed transitions

Take place for light nuclei such as ${}^3_1\text{H}, {}^{14}_6\text{C}, {}^{15}_6\text{C}, \dots$, where all protons and final neutrons at the Fermi level result from an overlap in the initial and final nuclear wave function. This means that the transitions are of the SP-type and yield maximum value of the F and GT matrix element.

2.9.2 l -forbidden allowed transitions

This type occurs in the case where simple SP transition in mean-field shell-model picture, forbidden by $\Delta l = 0$ selection rule included in eq.(16) and eq.(17) The selection rules in Table (2) are fulfilled. This means the forbiddingness is due to a single configuration approximate for Ψ_i and Ψ_f . Using a configuration mixing based on the residual interaction, such as the pairing effect, removes this forbiddingness and gives a finite value for $\log ft$, which is usually below 5 due to the lack of strength in the configuration mixing [23].

2.9.3 Unfavorable allowed transitions

Such transitions do not belong to either of the two types discussed above. They are allowed SP transitions in that there is no l forbiddingness. However, the SP transitions are suppressed in the Ψ_i and Ψ_f due to the residual interaction.

Table 4: Classification of β -decay transitions.

Type of transition	$\log ft$
Superaligned	2.9 – 3.7
Unfavoured allowed	3.8 – 6.7
l -forbidden allowed	≥ 5.0
First-forbidden unique	8 – 10
First-forbidden non-unique	6 – 9
Second-forbidden	11 – 13
Third-forbidden	17 – 19
Fourth-forbidden	> 22

3. Results and discussion

3.1. β -Decay transition in one-particle and one-hole nuclei

These are the simplest possible nuclei, which are considered stepping stone to more complex structure.

3.2. Matrix elements

The wave functions of one-particle and one-hole *nuclei* can be written as,

$$|\Psi_i\rangle = |n_i l_i j_i m_i\rangle = c_i^\dagger |CORE\rangle, \quad (34)$$

and

$$|\Psi_f\rangle = |n_f l_f j_f m_f\rangle = c_f^\dagger |CORE\rangle. \quad (35)$$

For the one-hole nuclei, the wave functions are,

$$|\Phi_i\rangle = |(n_i l_i m_i)^{-1}\rangle = h_i^\dagger |HF\rangle, \quad (36)$$

$$|\Phi_f\rangle = |(n_f l_f m_f)^{-1}\rangle = h_f^\dagger |HF\rangle.$$

The one-body transition densities, derived according to Fermi golden rule are

$$\langle\Psi_f \left\| \left[c_a^\dagger, \tilde{c}_b \right]_L \right\| \Psi_i\rangle = \hat{L} \delta_{af} \delta_{bi}, 0 \quad (37)$$

and

$$\langle\Phi_f \left\| \left[c_a^\dagger, \tilde{c}_b \right]_L \right\| \Phi_i\rangle = \hat{L} \delta_{ai} \delta_{bf} (-1)^{j_i + j_f + L}. \quad (38)$$

Here \hat{L} is a Wigner-Eckart normalization

$$\hat{L} = \frac{1}{\sqrt{2L+1}}, \quad (39)$$

where L is the resultant orbital angular momentum of the L_i and L_f coupling. Recall that

$$\left[c_a^\dagger, \tilde{c}_b \right]_L = \sum_{M_\alpha, M_\beta} \langle L_a M_\alpha L_b M_\beta | LM \rangle c_a^\dagger \tilde{c}_b |CORE\rangle, \quad (40)$$

and using (14) and (15) we have

$$\mathcal{M}_F = \langle \xi_f J_f \left\| \hat{1} \right\| \xi_i J_i \rangle = \delta_{J_i J_f} \sum_{a,b} \mathcal{M}_F(ab) \langle \xi_f J_f \left\| \left[c_a^\dagger \tilde{c}_b \right]_{\Delta J=0} \right\| \xi_i J_i \rangle, \quad (41)$$

and

$$\mathcal{M}_{GT} = \langle \xi_f J_f \| \hat{\sigma} \| \xi_i J_i \rangle = \sum_{a,b} \mathcal{M}_{GT}(ab) \left\langle \xi_f J_f \left\| \left[c_a^\dagger \tilde{c}_b \right]_{\Delta J=1} \right\| \xi_i J_i \right\rangle. \quad (42)$$

According to (18) and (19), there are symmetry relations between one-particle and one-hole amplitude,

$$\mathcal{M}_F(\Psi_i \rightarrow \Psi_f) = -\mathcal{M}_F(\Phi_i \rightarrow \Phi_f) = \delta_{if} \hat{J}_i, \quad (43)$$

and

$$\mathcal{M}_{GT}(\Psi_i \rightarrow \Psi_f) = \mathcal{M}_{GT}(\Phi_i \rightarrow \Phi_f) = \sqrt{3} \mathcal{M}_{GT}(ab), \quad (44)$$

where \mathcal{M}_F and $\mathcal{M}_{GT}(ab)$ are single particle (SP) Fermi and Gamow-Teller matrix elements, respectively. These are substituted into eq.(8) and eq.(9), which yields

$$B_F = \frac{g_v^2}{2J+1} |\mathcal{M}_F|^2, \quad (45)$$

and

$$B_{GT} = \frac{g_A^2}{2J+1} |\mathcal{M}_{GT}|^2, \quad (46)$$

where we obtain

$$B_F = g_v^2 \delta_{if}, \quad (47)$$

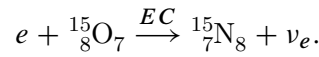
and

$$B_{GT} = g_A^2 \frac{3}{2J_i+1} |\mathcal{M}_{GT}(ab)|^2 = g_A^2 \frac{3}{2J_i+1} |\mathcal{M}_{GT}(fi)|^2. \quad (48)$$

The values are taken from references [8, 14]. The relation is valid for transitions between one-particle states and for transitions between one-hole states.

3.3. Calculating EC half-life for ^{15}O and ^{15}N isobars

The electronic capture equation is



We can calculate the Q -value as

$$Q_{EC} = 2.754 \text{ MeV}.$$

The experimental $\log ft$ value is 3.6 meaning the transition is superallowed. The decay scheme and experimental information are shown in fig.(4).

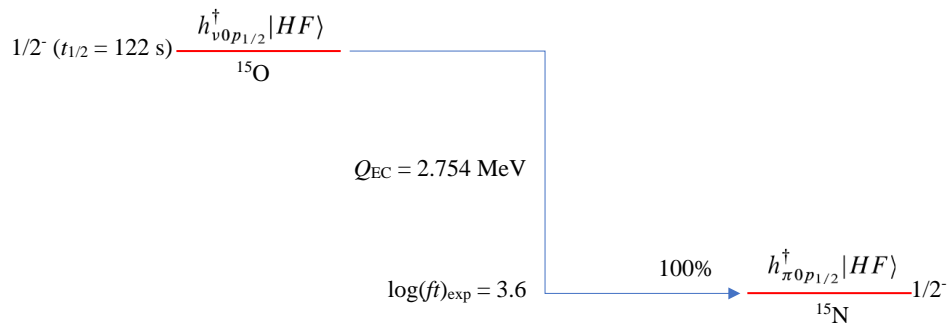


Figure 4: The decay scheme of ^{15}O in ground state to ^{15}N ground state via the β^+/EC decay mode. The experimental half-life, Q value, branching and $\log(ft)$ value are shown in the figure.

The Fermi SP-matrix is

$$\mathcal{M}_F(ab) = \langle f \| \hat{1} \| i \rangle = \delta_{fi} \hat{J}_a = \langle n_f l_f j_f \| 1 \| n_i l_i j_i \rangle = \delta_{n_f n_i} \delta_{l_f l_i} \delta_{j_f j_i} J_b. \quad (49)$$

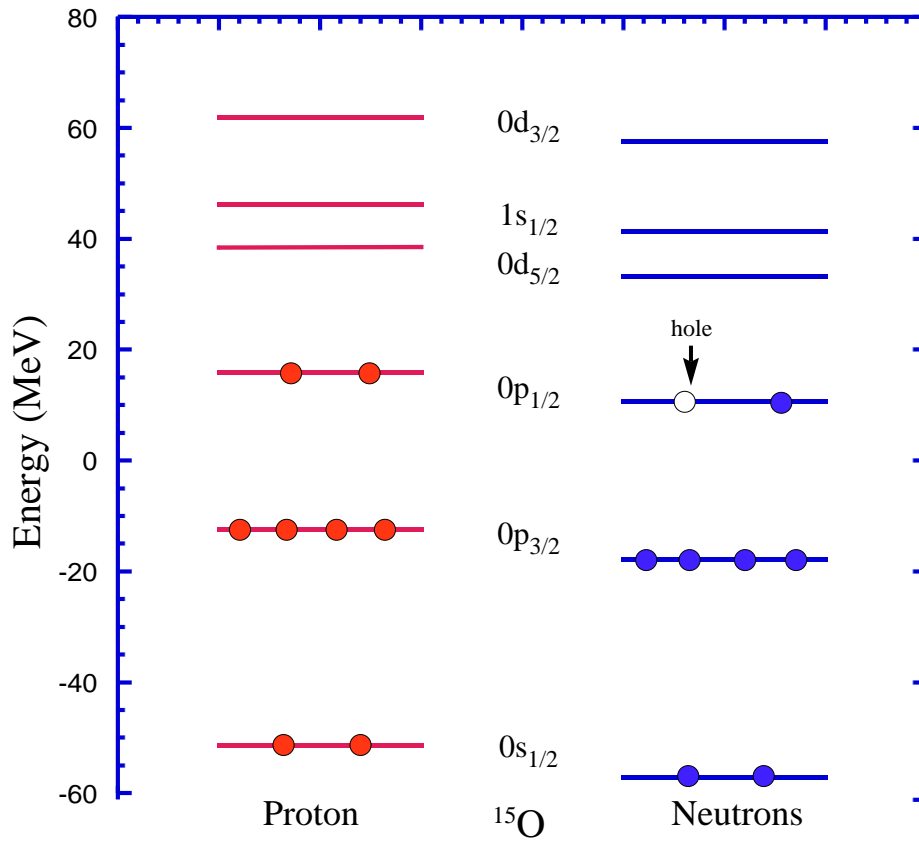


Figure 5: Single particle states of ^{15}O generated using JISP6 potential in 8-shell model space. The SP particle transition $\pi 0p_{1/2} \rightarrow \nu 0p_{1/2}$ energy is $\Delta E = 5.221$ MeV.

Using eq.(47), the reduced Fermi matrix is thus

$$B_F = g_V^2 = 1.0, \quad (50)$$

The GT SP-matrix is

$$\begin{aligned} \mathcal{M}_{GT}(fi) &= \frac{1}{\sqrt{3}} \langle f \| \hat{\sigma} \| i \rangle = \frac{1}{\sqrt{3}} \langle n_f l_f j_f \| \sigma \| n_i l_i j_i \rangle \\ &= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \times 2 \delta_{n_f n_i} \delta_{l_f l_i} \hat{j}_f \hat{j}_i (-1)^{l_f + j_i + \frac{3}{2}} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ j_f & j_i & L_f \end{array} \right\} \\ &= -\sqrt{\frac{2}{3}}. \end{aligned} \quad (51)$$

The $6j$ symbol is calculated using Mathematica. Make use eq.(51) into eq.(9) the reduced GT transition matrix is thus

$$B_{GT} = g_A^2 \frac{3}{2} \left| \mathcal{M}_{GT}(\pi 0 p_{\frac{1}{2}} \rightarrow \nu 0 p_{\frac{1}{2}}) \right|^2 = (1.25)^2 \times \frac{3}{2} \left(-\sqrt{\frac{2}{3}} \right)^2 = 0.521. \quad (52)$$

Make use eq.(50) and eq.(52) into eq.(6), we obtain

$$f_0 t_{\frac{1}{2}} = \frac{6147s}{B_F + B_{GT}} = \frac{6147}{1.0 + 0.521} = 4041.4 \text{ s}. \quad (53)$$

Using the experimental half-life $t_{1/2} = 122$ s, we can obtain the f_0 using eq.(6) which yields

$$f_0 = \frac{4041}{122} = 33.12. \quad (54)$$

This corresponds to $\log(ft)$ value

$$\log(ft) = 3.61, \quad (55)$$

as shown experimentally. The f_0 phase space function can be broken into two phase space functions: one for β^+ -decay and the other for EC decay. Therefore,

$$f_0 = f_0^{(+)} + f_0^{EC}. \quad (56)$$

The nuclear energy difference in eq.(22) is

$$E_0 = \frac{Q_{EC^-} - m_e c^2}{m_e c^2} = \frac{2.754 - 0.511 \text{ Mev}}{0.511 \text{ Mev}} = 4.389. \quad (57)$$

We can use Primakoff-Rosen approximation eq.(27) [21]

$$F_o^{(PR)(\pm)}(\mp Z_f) = \frac{2\pi\alpha Z_f}{1 - e^{-2\pi\alpha Z_f}} = \frac{2\pi \frac{1}{137}(7)}{1 - e^{-2\pi \frac{1}{137}(7)}} = 0.848, \quad (58)$$

with the aid of the phase space expansion (28) we find $f_0^{(+)}$

$$f_0^{(+)} = 42.3, \quad (59)$$

and use eq.(24) we get

$$f_0^{EC} = 0.036. \quad (60)$$

We notice that $f_0^{EC} \ll f_0^{(+)}$, which indicates that the transition is dominated by β^+ . The total value of the phase factor becomes

$$f_0 = f_0^{EC} + f_0^{(+)} = 42.336, \quad (61)$$

Hence

$$\log(f_0) = 1.63, \quad (62)$$

using the value of $\log(ft)$ in eq.(55), we can calculate the decay half-life

$$t_{\frac{1}{2}} = 10^{(\log ft - \log f_0)} = 10^{(3.61 - 1.63)} = 95.5 \text{ sec}, \quad (63)$$

This deviate from the experimental half-life ($t_{1/2} = 122 \text{ sec}$). The previous steps for calculating the half-lives for one-particle-hole isotopes are coded using C++ code-named "beta_decay_halfife_oneph.cpp" to generate the data shown in table (5).

Table 5: Half-lives for odd-even (even-odd) isotopes computed using one-particle-hole technique. The computed values are compared with the experimental results.

Beta Decay	$Q_{EC}^{(\text{exp})}$ (MeV)	$\log(f_0)$	$\log(ft)$	$t_{\frac{1}{2}}$ (s)	$t_{\frac{1}{2}}^{(\text{exp})}$ (s)
$^{15}\text{O} (1/2^+) \rightarrow ^{15}\text{N} (1/2^-)$	2.754	1.626	3.606	95.5	122
$^{17}\text{F} (5/2^+) \rightarrow ^{17}\text{O} (5/2^+)$	2.762	1.624	3.283	45.6	64.5
$^{39}\text{Ca} (3/2^+) \rightarrow ^{39}\text{K} (3/2^+)$	6.524	3.671	3.500	0.675	0.86
$^{41}\text{Sc} (7/2^-) \rightarrow ^{41}\text{Ca} (7/2^-)$	6.495	6.495	3.308	0.456	0.59

3.4. Beta decay to and from the even-even ground state

Charge-changing excitations of particle-hole nuclei can undergo beta decay to the reference nucleus. The initial state is an odd-odd nucleus, generated by making charge-changing particle-hole excitations of the particle-hole vacuum. Let the final state be the particle-hole vacuum $|HF\rangle$ is the ground state for the reference nucleus. Particle-hole excited states are created by letting one nucleon jump from a state below the Fermi level to a state above it [24]. The wave functions of particle-hole nuclei is

$$|ab^{-1}; JM\rangle = [c_a^\dagger h_b^\dagger]_{JM} |HF\rangle = [c_a^\dagger \tilde{c}_b]_{JM} |HF\rangle, \quad (64)$$

The wave functions are normalized

$$\langle ab^{-1}; JM | cd^{-1}; J'M'\rangle = \delta_{ac}\delta_{bd}\delta_{JJ'}\delta_{MM'}. \quad (65)$$

Using (12), with the aid of eq.(64) and normalization (65), The β -decay matrix elements are constructed from the transition density

$$\begin{aligned} \langle HF | \left\| [c_a^\dagger \tilde{c}_b]_L \right\| a_i b_i^{-1}; J_i \rangle &= \frac{\delta_{LJ_i} \delta_{ab_i} \delta_{ba_i} (-1)_i^{j_{a_i} - j_{b_i} + J_i}}{\sqrt{2J_i + 1}} \\ &= \delta_{LJ_i} \delta_{ab_i} \delta_{ba_i} (-1)_i^{j_{a_i} - j_{b_i} + J_i} \hat{J}_i \end{aligned} \quad (66)$$

Inserting eq.(41) and eq.(42), yields

$$\mathcal{M}_F(a_i b_i^{-1}) = \delta_{J_i 0} \delta_{a_i b_i} \hat{J}_{a_i}, \quad (67)$$

and

$$\mathcal{M}_{GT}(a_i b_i^{-1}) = -\sqrt{3} \delta_{J_i 1} \mathcal{M}_{GT}(a_i b_i), \quad (68)$$

where the symmetry relations (18) and (19) are used. In the case of the odd-odd nucleus has a low-lying state below the particle-hole vacuum of the reference nucleus, β -decay occur from vacuum to the odd-odd nucleus. This is common in light nuclei. Therefore eq.(66) is now replaced by

$$\langle a_f b_f^{-1}; J_i | \left\| [c_a^\dagger \tilde{c}_b]_L \right\| HF \rangle = \delta_{LJ_f} \delta_{aa_f} \delta_{bb_f} \hat{J}_f. \quad (69)$$

Substituting eq.(41) and eq.(42), yields

$$\mathcal{M}_F(a_f b_f^{-1}) = \delta_{J_f 0} \delta_{a_f b_f} \hat{J}_{a_f}, \quad (70)$$

and

$$\mathcal{M}_{GT}(a_f b_f^{-1}) = -\sqrt{3} \delta_{J_f 1} \mathcal{M}_{GT}(a_f b_f). \quad (71)$$

3.5. Calculating the strength function of ^{56}Ni decay

The β^+ / EC of



The ^{56}Ni is in the ground state 0^+ , whereas ^{56}Co can be formed at excited states. The excited states of ^{56}Co is shown in fig.(6). The Q -value of the decay is $Q_{EC} = 2.13$ MeV. This means the possible excited states ^{56}Co can form at 2.06 MeV. This includes states 1^+ , 2^+ , 3^+ , 4^+ , 5^+ , and 6^+ .

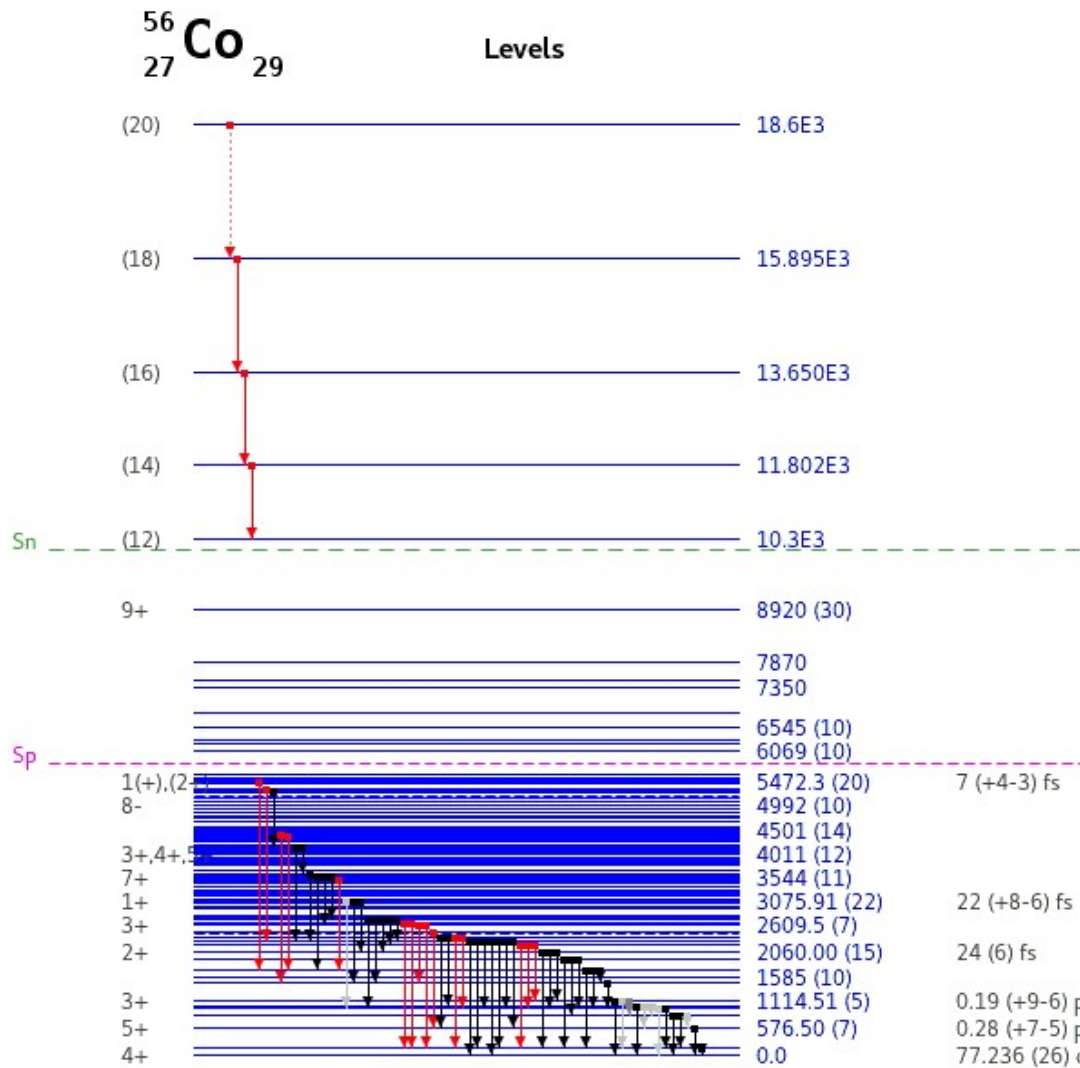


Figure 6: Energy levels of ^{56}Co . The ^{56}Co is the by-products of the ^{56}Ni EC decay with $Q_{EC} = 2.13$ MeV. Thus ^{56}Co is formed in states with energy less than $E_x \leq 2.06$ MeV. Taken from International Atomic Energy Agency.

In the SP scheme, the valence of ^{56}Ni in the ground state is shown in fig.(7). It shows that the core $|\text{CORE}\rangle_\pi$ is filled with 20 protons and $|\text{CORE}\rangle_\nu$ is filled with 20 neutrons. The valence states $|\text{HF}\rangle_\pi$ and $|\text{HF}\rangle_\nu$ has the $0f_{7/2}$ state full with 8 protons and 8 neutrons.

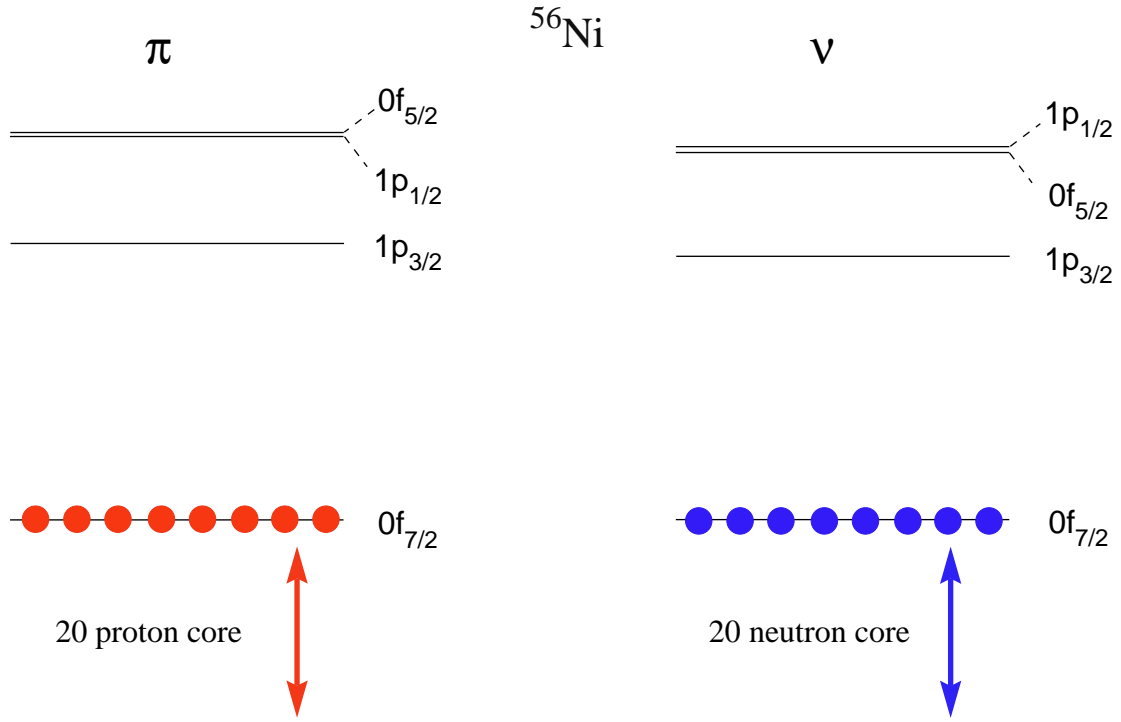


Figure 7: Valence shells $|HF\rangle$ state for ^{56}Ni . At ground state, only $0f_{7/2}$ states are full of protons and neutrons. The state spacings are proportional to the energy spacings generated by Wood-Saxon potential in reference [25].

When one proton in the $\pi 0f_{7/2}$ state of ^{56}Ni captures an electron this causes vacancy in the $\pi 0f_{7/2}$ state of the daughter ^{56}Co and an extra neutron is formed. Only this transition

$$\pi 0f_{7/2} \rightarrow \nu 0f_{5/2},$$

is allowed by the SP transition matrix which set the selection rule for either Fermi or GT transition, given in eq.(16) and (17). Thus, the extra neutron must be formed in the $\nu 0f_{5/2}$ state of ^{56}Co . In another word, a proton-hole created in the $\pi 0f_{7/2}$ and a neutron-particle is created in the $\nu 0f_{5/2}$. This is shown in fig.(8).

The *triangular condition* $\Delta(J \frac{5}{2} \frac{7}{2})$ gives all possible nuclear spin states of ^{56}Co , thus

$$\left| \frac{7}{2} - \frac{5}{2} \right| = 1 \leq J \leq \left| \frac{7}{2} + \frac{5}{2} \right| = 6.$$

Henceforth, the nuclear state of ^{56}Co is defined in terms of the ground state of the nuclear state of ^{56}Ni as

$$|^{56}\text{Co}, 1^+, 2^+, 3^+, 4^+, 6^+\rangle = \left[c_{\nu 0f_{5/2}}^\dagger h_{\pi 0f_{7/2}}^\dagger \right]_{1^+, 2^+, 3^+, 4^+, 5^+, 6^+} |^{56}\text{Ni}; 0^+\rangle. \quad (73)$$

Since $\delta_{J_f 0} = 0 \Rightarrow \mathcal{M}_F = 0$ in eq.(70), only possible final state for the Gamow-Teller matrix is the 1^+ state of ^{56}Co according to eq.(71). Thus the GT SP transition matrix according to eq.(71) is

$$\mathcal{M}_{GT} = \sqrt{3} \mathcal{M}_{GT}(f_{\frac{5}{2}} f_{\frac{7}{2}}) = -\sqrt{3}(-4\sqrt{\frac{2}{7}}) = 3.703, \quad (74)$$

thus,

$$B_F = 0. \quad (75)$$

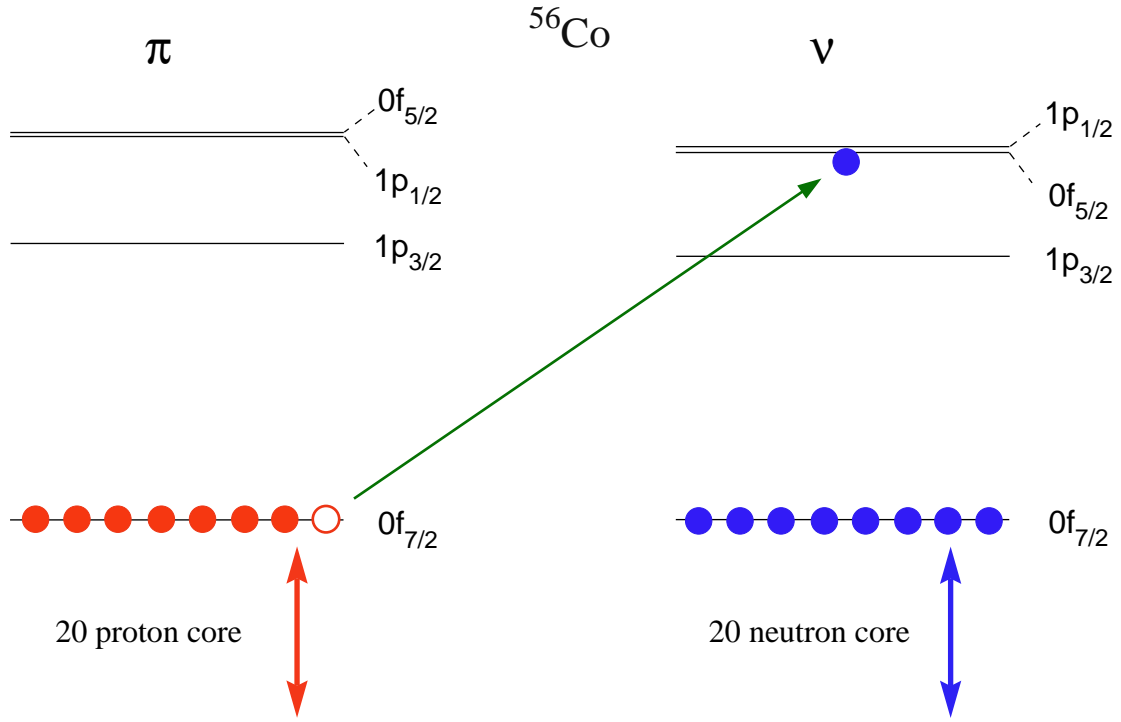


Figure 8: Valence shells $|HF\rangle$ state for ^{56}Co . A proton hole p^{-1} is created in the $\pi 0f_{7/2}$ state ($h_{\pi 0f_{7/2}}^\dagger$) and a neutron-particle is created in the $\nu 0f_{5/2}$ state ($c_{\nu 0f_{5/2}}^\dagger$). The state spacings are proportional to the energy spacings generated by Wood-Saxon potential in reference [25].

Make use of eq.(48) into eq.(74) the reduced GT transition matrix is thus,

$$B_{GT} = g_A^2 \frac{3}{2J_i + 1} |\mathcal{M}_{GT}(fi)|^2 = (1.25)^2 \frac{3}{(2 \times 1) + 1} (3.703)^2 = 21.43. \quad (76)$$

Make use of eq.(75) and eq.(76) into eq.(6), we obtain

$$f_0 t_{\frac{1}{2}} = \frac{\kappa}{(B_F + B_{GT})} = \frac{6147}{21.43} = 286.84. \quad (77)$$

Using eq.(11) and eq.(77), we obtain,

$$\log ft = \log 286.84 = 2.46, \quad (78)$$

The experimental value,

$$(\log ft)_{\text{exp}} = 4.4. \quad (79)$$

This is much less than the experimental value $\log ft = 4.4$. An indication for the failure of the particle-hole theory. According to table (4), the transition is unfavoured allowed, meaning that the single-particle transition is suppressed during the initial and the final states due to the residual two-body interaction by a factor $10^{2.46-4.4} = 0.011$.

The coupling in the excitation

$$|^{56}\text{Co}; 1^+\rangle = \left[c_{\nu 0f_{5/2}}^\dagger h_{\pi 0f_{7/2}}^\dagger \right] |^{56}\text{Ni}; 0^+\rangle,$$

is the only way to produce 1^+ state by exciting a proton from $0f_{7/2}$ shell to the $0f_{5/2}$ for neutrons. Therefore, The discrepancy suggests the need for better configurations consists of two-particle-hole excitations which play an active part in the low laying states of ^{56}Co .

3.6. β -decay transitions between two particle-hole states

This configuration is based on the electromagnetic transitions between two arbitrary particle-hole states:

$$|a_i b_i^{-1}; J_i\rangle \rightarrow |a_f b_f^{-1}; J_f\rangle,$$

due to an operator \hat{O}_L in which the transition amplitude is [26, 27]

$$\begin{aligned} \langle a_f b_f^{-1}; J_f | \hat{O}_L | a_i b_i^{-1}; J_i \rangle &= (-1)^{j_{a_i} + j_{b_f}} \hat{J}_i \hat{J}_f \times \\ &\left[\begin{array}{l} \delta_{b_i b_f} (-1)^{J_i + L} \left\{ \begin{array}{ccc} J_i & J_f & L \\ j_{a_f} & j_{a_i} & j_{b_i} \end{array} \right\} \langle a_i | \hat{O}_L | a_f \rangle + \\ \delta_{a_i a_f} (-1)^{J_f + L} \left\{ \begin{array}{ccc} J_i & J_f & L \\ j_{b_f} & j_{b_i} & j_{a_i} \end{array} \right\} \langle b_i | \hat{O}_L | b_f \rangle \end{array} \right]. \end{aligned} \quad (80)$$

Here the a 's and the b 's are replaced by π 's and ν 's.

Starting from even-even reference nucleus ($N - Z$), has a particle-hole vacuum state. The excited states are proton-particle-hole (pp^{-1}) and neutron-particle-hole (nn^{-1}) excitations. Consider β^- decay of the adjacent odd-odd ($Z + 1, N - 1$) we have (np^{-1}) excitations. Let us define β^- decay operator $\hat{\beta}_{LM}^-$ as

$$\hat{\beta}_{LM}^- = (\hat{L})^{-1} \sum_{pn} \langle p | \hat{\beta}_L | n \rangle [c_p^\dagger \tilde{c}_n]_{LM} \quad (81)$$

for $L = 0$ is Fermi operator and for $L = 1$ Gamow-Teller operator. Thus $\beta_0 = 1$ and $\beta_1 = \sigma$. The transition amplitude for state $|\Psi_i\rangle$ to $|\Psi_f\rangle$ is given by

$$\langle \Psi_f | \hat{\beta}_L^- | \Psi_i \rangle = (\hat{L})^{-1} \sum_{pn} \langle p | \beta_L | n \rangle \langle \Psi_f | [c_p^\dagger \tilde{c}_n] | \Psi_i \rangle. \quad (82)$$

We have the following different β -decay cases

3.6.1 The initial state is a neutron-particle-proton-hole ($n_i p_i^{-1}$)

The initial nuclear wave function is

$$|\Psi_i\rangle = [c_{n_i}^\dagger h_{p_i}^\dagger]_{J_i M_i} |HF\rangle. \quad (83)$$

For the final nuclear wave function we need to consider two sub-cases: either the final state be neutron-particle-neutron-hole ($n_f n_f'^{-1}$) or proton-particle-proton-hole ($p_f p_f'^{-1}$).

Neutron-particle-neutron-hole final state ($n_f n_f'^{-1}$) Here the final nuclear wave function is

$$|\Psi_f\rangle = [c_{n_f}^\dagger h_{n_f'}^\dagger]_{J_f M_f} |HF\rangle. \quad (84)$$

Make use into (83) (84) into (82) use Wigner-Eckart theorem

$$\begin{aligned} \langle n_f n_f'^{-1}; J_f | \hat{\beta}_L^- | n_i p_i^{-1}; J_i \rangle &= \delta_{n_f n_i} (-1)^{j_{n_i} + j_{n_f'} + J_f + 1} \hat{J}_i \hat{J}_f \hat{L} \times \\ &\left\{ \begin{array}{ccc} J_i & J_f & L \\ j_{n_f'} & j_{p_i} & j_{n_i} \end{array} \right\} \mathcal{M}_L(p_i n_f'). \end{aligned} \quad (85)$$

Note that n_f' represents neutron hole n^{-1} . $\mathcal{M}_L(p_i n_f')$ is either the Fermi ($L = 0$) or the Gamow-Teller ($L = 1$) single-particle matrix element given in 14 and 15, respectively.

$$\mathcal{M}_0(ab) = \mathcal{M}_F(ab) = \langle a | \beta_0 | b \rangle = \langle a | 1 | b \rangle, \quad (86)$$

$$\mathcal{M}_1(ab) = \mathcal{M}_{GT}(ab) = \frac{1}{\sqrt{3}} \langle a | \beta_1 | b \rangle = \frac{1}{\sqrt{3}} \langle a | \sigma | b \rangle. \quad (87)$$

In the single-particle matrix element \mathcal{M}_L proton and neutrons labels are no longer distinct.

Proton-particle-proton-hole final state ($p_f p_f'^{-1}$) The final wave function

$$|\Psi_f\rangle = \left[c_{p_f}^\dagger h_{p_f'}^\dagger \right]_{J_f M_f} |HF\rangle. \quad (88)$$

Make use into eq (82), we obtain

$$\begin{aligned} \left\langle p_f p_f'^{-1}; J_f \parallel \beta_L^- \parallel n_i p_i^{-1}; J_i \right\rangle &= \delta_{p_i p_f'} (-1)^{j_{n_i} + j_{p_f'} + J_i + L} \hat{J}_i \hat{J}_f \hat{L} \times \\ &\left\{ \begin{matrix} J_i & J_f & L \\ j_{p_f} & j_{n_i} & j_{p_i} \end{matrix} \right\} \mathcal{M}_L(n_i p_f). \end{aligned} \quad (89)$$

For β^+ decay, initial state odd-odd ($Z + 1, N - 1$) nucleus generated by proton- particle-neutron-hole (pn^{-1}) excitation of (N, Z) particle-hole vacuum. The final state is particle-hole excitation, which obeys the charge conservation condition, for the even-even reference nucleus (Z, N). In similar argument to eq (81), the decay operator is:

$$\beta_{LM}^{+-} = \sum_{pn} \langle n \parallel \beta_L \parallel p \rangle \left[c_n^\dagger \tilde{c}_p \right]_{LM}. \quad (90)$$

Thus the transition amplitude is similar to eq.(82), which reads as

$$\langle \Psi_f \parallel \beta_L^{+-} \parallel \Psi_i \rangle = (\hat{L})^{-1} \sum_{pn} \langle n \parallel \beta_L \parallel p \rangle \langle \Psi_f \parallel [c_n^\dagger \tilde{c}_p] \parallel \Psi_i \rangle. \quad (91)$$

3.6.2 The initial state is a proton-particle-neutron-hole state ($p_i n_i^{-1}$)

In this case, the initial nuclear wave function is

$$|\Psi_i\rangle = \left[c_{p_i}^\dagger h_{n_i}^\dagger \right]_{J_i M_i} |HF\rangle. \quad (92)$$

Again, to obtain the final nuclear wave function we need to consider two sub-cases: either the final state be neutron-particle-neutron-hole ($n_f n_f'^{-1}$) or proton-particle-proton-hole ($p_f p_f'^{-1}$).

Neutron-particle-neutron-hole final state ($n_f n_f'^{-1}$) In this case, the final nuclear wave function becomes

$$|\Psi_f\rangle = \left[c_{n_f}^\dagger h_{n_f'}^\dagger \right]_{J_f M_f} |HF\rangle. \quad (93)$$

The transition amplitude becomes

$$\begin{aligned} \left\langle n_f n_f'^{-1}; J_f \parallel \beta_L^+ \parallel p_i n_i^{-1}; J_i \right\rangle &= \delta_{n_i n_f'} (-1)^{j_{p_i} + j_{n_f'} + J_i + L} \hat{J}_i \hat{J}_f \hat{L} \times \\ &\left\{ \begin{matrix} J_i & J_f & L \\ j_{n_f} & j_{p_i} & j_{n_i} \end{matrix} \right\} \mathcal{M}_L(p_i n_i). \end{aligned} \quad (94)$$

Proton-particle-proton-hole final state ($p_f p_f'^{-1}$) The final nuclear wave function is

$$|\Psi_f\rangle = \left[c_{p_f}^\dagger h_{p_f'}^\dagger \right]_{J_f M_f} |HF\rangle. \quad (95)$$

The transition amplitude becomes:

$$\begin{aligned} \left\langle p_f p_f'^{-1}; J_f \parallel \beta_L^+ \parallel p_i n_i^{-1}; J_i \right\rangle &= \delta_{p_i p_f} (-1)^{j_{p_i} + j_{p_f'} + J_f + 1} \hat{J}_i \hat{J}_f \hat{L} \times \\ &\left\{ \begin{matrix} J_i & J_f & L \\ j_{p_f'} & j_{n_i} & j_{p_i} \end{matrix} \right\} \mathcal{M}_L(n_i p_f'). \end{aligned} \quad (96)$$

The allowed transitions for Fermi ($L = 0$) decay types, in eqs.(85), (89), (94), and (96), can be written as [14]

$$\begin{aligned} \left(n_f n_f'^{-1}; J_f \parallel \beta_F^- \parallel n_i p_i^{-1}; J_i \right) &= \delta_{n_f n_i} (-1)^{j_{n_i} + j_{n_f'} + J_f + 1} \hat{J}_i \hat{J}_f \left\{ \begin{matrix} J_i & J_f & 0 \\ j_{n_f'} & j_{p_i} & j_{n_i} \end{matrix} \right\} \mathcal{M}_L(p_i n_f') \\ &= -\delta_{n_f n_i} \delta_{J_i J_f} \delta_{p_i n_f'} \hat{J}_i \Delta(j_{n_i} j_{n_f'} J_i), \end{aligned} \quad (97)$$

$$\begin{aligned} \left(p_f p_f'^{-1}; J_f \parallel \beta_F^- \parallel n_i p_i^{-1}; J_i \right) &= \delta_{p_i p_f'} (-1)^{j_{n_i} + j_{p_f'} + J_i} \hat{J}_i \hat{J}_f \left\{ \begin{matrix} J_i & J_f & 0 \\ j_{p_f} & j_{n_i} & j_{p_i} \end{matrix} \right\} \mathcal{M}_L(n_i p_f), \\ &= \delta_{p_i p_f'} \delta_{J_i J_f} \delta_{n_i p_f} \hat{J}_i \Delta(j_{p_i} j_{p_f} J_i). \end{aligned} \quad (98)$$

$$\begin{aligned} \left(n_f n_f'^{-1}; J_f \parallel \beta_F^+ \parallel p_i n_i^{-1}; J_i \right) &= \delta_{n_i n_f'} (-1)^{j_{p_i} + j_{n_f'} + J_i} \hat{J}_i \hat{J}_f \left\{ \begin{matrix} J_i & J_f & 0 \\ j_{n_f} & j_{p_i} & j_{n_i} \end{matrix} \right\} \mathcal{M}_L(p_i n_i), \\ &= \delta_{n_i n_f'} \delta_{J_i J_f} \delta_{p_i n_i} \hat{J}_i \Delta(j_{n_i} j_{n_f} J_i). \end{aligned} \quad (99)$$

$$\begin{aligned} \left(p_f p_f'^{-1}; J_f \parallel \beta_F^+ \parallel p_i n_i^{-1}; J_i \right) &= \delta_{p_i p_f} (-1)^{j_{p_i} + j_{p_f'} + J_f + 1} \hat{J}_i \hat{J}_f \left\{ \begin{matrix} J_i & J_f & 0 \\ j_{p_f'} & j_{n_i} & j_{p_i} \end{matrix} \right\} \mathcal{M}_L(n_i p_f'), \\ &= -\delta_{p_i p_f} \delta_{J_i J_f} \delta_{n_i p_f'} \hat{J}_i \Delta(j_{p_i} j_{p_f'} J_i). \end{aligned} \quad (100)$$

Here $\hat{j} = \sqrt{2j+1}$, and δ_{pn} indicates that the quantum numbers of the proton and neutron orbitals have to be the same. The symbol $\Delta(j_1 j_2 j)$ denotes the *triangular condition*

$$|j_1 - j_2| \leq j \leq |j_1 + j_2|. \quad (101)$$

3.7. Calculating β^- -decay strength ft for ^{16}N

The β^- -decay equation is



We can calculate the Q -value

$$Q_{\beta^-} = 10.419 \text{ MeV}. \quad (103)$$

The allowed transition,

1. $\nu 0p_{\frac{1}{2}} \rightarrow \pi 0p_{\frac{1}{2}} \implies (\nu 0p_{\frac{1}{2}})^{-1} (\nu 0d_{\frac{5}{2}})$
2. $\nu 0d_{\frac{5}{2}} \rightarrow \pi 0p_{\frac{1}{2}} \implies 0^+$
3. $\nu 0d_{\frac{5}{2}} \rightarrow \pi 0d_{\frac{5}{2}} \implies (\pi 0d_{\frac{5}{2}}) (\nu 0p_{\frac{1}{2}})^{-1}$
4. $\nu 0p_{\frac{1}{2}} \rightarrow \pi 0p_{\frac{1}{2}} \ \& \ (\nu 0d_{\frac{5}{2}}) \rightarrow (\nu 1s_{\frac{1}{2}}) \implies \nu 1s_{\frac{1}{2}} (\nu 0p_{\frac{1}{2}})^{-1}$
5. $\nu 0d_{\frac{5}{2}} \rightarrow \pi 1s_{\frac{1}{2}} \implies (\pi 1s_{\frac{1}{2}}) (\pi 0p_{\frac{1}{2}})^{-1}$

Note that: the selection rule prohibits transition among d -states. There is no transition from $0p_{\frac{1}{2}} \rightarrow 1s_{\frac{1}{2}}$ because the Q -value of the decay is smaller than the energy gap.

The initial state is neutron-particle-proton hole ($n_i p_i^{-1}$) in eq.(83). Let the final state be a neutron-particle-neutron hole or proton-particle-proton hole this,

$$\Psi_f = c_{\pi}^{\dagger} h_{\pi}^{\dagger} |HF\rangle = \left| p_f p_f^{-1} \right\rangle \text{ or } \Psi_f = c_{\nu}^{\dagger} h_{\nu}^{\dagger} |HF\rangle = \left| n_f n_f^{-1} \right\rangle. \quad (104)$$

Let us take the transitions one by one:

1- $\nu 0p_{\frac{1}{2}} \rightarrow (\pi 0p_{\frac{1}{2}}) \Rightarrow \nu 0d_{\frac{5}{2}}(\nu 0p_{\frac{1}{2}})^{-1}$: Using the SP states shown in fig.(9), this transition is energy absorbing transition $\Delta E = -4$ MeV. Using eq.(85), the Gamow-Teller decay amplitude becomes

$$\begin{aligned} \left\langle \nu 0d_{\frac{5}{2}}(\nu 0p_{\frac{1}{2}})^{-1}; J_f \parallel B_{GT}^- \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \right\rangle &= (-1)^{\frac{5}{2}+\frac{1}{2}+J_f+1} \hat{J}_f \times \\ \sqrt{2 \times 1 + 1} \sqrt{2 \times 1 + 1} \left\{ \begin{matrix} 2 & J_f & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{matrix} \right\} \mathcal{M}_L(0p_{\frac{1}{2}} \rightarrow 0p_{\frac{1}{2}}), & \quad (105) \end{aligned}$$

$$\begin{aligned} &= (-1)^{J_f} \sqrt{15} \hat{J}_f \left\{ \begin{matrix} 2 & J_f & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{matrix} \right\} \left(-\frac{\sqrt{2}}{3}\right), \\ &= (-1)^{J_f+1} \sqrt{\frac{10}{3}} \hat{J}_f \left\{ \begin{matrix} 2 & J_f & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{matrix} \right\}, \\ &= A_1(J_f). \end{aligned} \quad (106)$$

The possible values of J_f can be obtained using the triangular condition

$$\left(\frac{5}{2} J_f\right) \Rightarrow J_f = \{1, 2, 3\}.$$

The $J_f = 1$ amplitude returns null because of the $6j$ -symbol in eq.(105). Only $J_f = \{2, 3\}$ are allowed. Thus

$$\left\langle \nu 0d_{\frac{5}{2}}(\nu 0p_{\frac{1}{2}})^{-1}; J_f \parallel \beta_{GT}^- \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \right\rangle = A_1(J_f). \quad (107)$$

Performing the calculation we find

$$A_1(2) = 0.608581 \text{ and } A_1(3) = 1.13855.$$

For Fermi transition, we must have $J_f = J_i$ according to the selection rule. Thus the Fermi decay amplitude becomes

$$\left\langle \nu 0d_{\frac{5}{2}}(\nu 0p_{\frac{1}{2}})^{-1}; 2^- \parallel \beta_F^- \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \right\rangle = -\sqrt{5} \quad (108)$$

2- $\nu 0d_{\frac{5}{2}} \rightarrow (\pi 0d_{\frac{5}{2}}) \Rightarrow \pi 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}$: This is energy absorbing transition $\Delta E = -3.5$ MeV. Using eq.(89), the GT decay amplitude becomes,

$$\begin{aligned} \left\langle \pi 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; J_f \parallel B_{GT} \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \right\rangle &= (-1)^{\frac{5}{2}+\frac{1}{2}+2+1} \hat{J}_f \times \\ \sqrt{2 \times 2 + 1} \sqrt{2 \times 1 + 1} \left\{ \begin{matrix} 2 & J_f & 1 \\ \frac{5}{2} & \frac{5}{2} & \frac{1}{2} \end{matrix} \right\} \mathcal{M}_L(d_{\frac{5}{2}} \rightarrow d_{\frac{5}{2}}), & \quad (109) \end{aligned}$$

$$\begin{aligned} &= \sqrt{42} \hat{J}_f \times \\ &\quad \left\{ \begin{matrix} 2 & J_f & 1 \\ \frac{5}{2} & \frac{5}{2} & \frac{1}{2} \end{matrix} \right\}, \\ &= A_2(J_f). \end{aligned} \quad (110)$$

For Fermi transition, we must have $J_f = J_i$ according to the selection rule. Thus the Fermi decay amplitude becomes

$$\left\langle \pi 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \parallel \beta_F^- \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \right\rangle = +\sqrt{5} \quad (111)$$

3- The transition $\nu 0d_{\frac{5}{2}} \rightarrow \pi 0p_{\frac{1}{2}} \Rightarrow 0^+$: yields $\langle HF; 0^+ \parallel \beta_{GT} \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2^- \rangle$ is not explainable using particle-hole theory.

4- $\nu 0p_{\frac{1}{2}} \rightarrow \pi 0p_{\frac{1}{2}}$ and $\nu 0d_{\frac{1}{2}} \rightarrow \nu 1s_{\frac{1}{2}} \Rightarrow \nu 1s_{\frac{1}{2}}(\nu 0p_{\frac{1}{2}})^{-1}$: Using eq.(85) the GT decay amplitude becomes,

$$\left\langle \nu 1s_{\frac{1}{2}}(\nu 0p_{\frac{1}{2}})^{-1}; J_f \parallel \beta_{GT} \parallel \nu 0d_{\frac{1}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2 \right\rangle = 0, \quad (112)$$

because neutron final particle ($\nu 1s_{\frac{1}{2}}$) does not match the neutron in initial particle ($\nu 0d_{\frac{1}{2}}$).

5- $\nu 0d_{\frac{5}{2}} \rightarrow \pi 1s_{\frac{1}{2}} \Rightarrow (\pi 1s_{\frac{1}{2}})(\pi 0p_{\frac{1}{2}})^{-1}$: Using eq.(89), for GT transition, the GT decay amplitude becomes

$$\left\langle \pi 1s_{\frac{1}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; J_f \parallel \beta_{GT} \parallel \nu 0d_{\frac{5}{2}}(\pi 0p_{\frac{1}{2}})^{-1}; 2 \right\rangle = 0, \quad (113)$$

because initial and final orbital angular momenta do not match.

Using eq.(107) and eq.(110),

$$\begin{aligned} \langle 2^- \parallel \beta_{GT} \parallel 2_{gs}^- \rangle &= \frac{A_1(2) + A_2(2)}{\sqrt{2}} = \frac{0.608581 + 2.55604}{\sqrt{2}} = 2.23772, \\ \langle 3^- \parallel \beta_{GT} \parallel 2_{gs}^- \rangle &= \frac{A_1(3) + A_2(3)}{\sqrt{2}} = \frac{(1.13855 + 0.57735)}{\sqrt{2}} = 1.28812. \end{aligned} \quad (114)$$

Make use eq.(114) and eq.(9), obtain

$$\begin{aligned} B_{GT} \langle 2_{gs}^- \rightarrow 2^- \rangle &= \frac{g_A^2}{2J_i + 1} |\langle 2^- \parallel \beta_{GT} \parallel 2_{gs}^- \rangle|^2 = 1.56481, \\ B_{GT} \langle 2_{gs}^- \rightarrow 3^- \rangle &= \frac{g_A^2}{2J_i + 1} |\langle 3^- \parallel \beta_{GT} \parallel 2_{gs}^- \rangle|^2 = 0.518517. \end{aligned} \quad (115)$$

Make use eq.(115) into eq.(6),eq.(11), we obtain,

$$\begin{aligned} \log ft(2_{gs}^- \rightarrow 2^-) &= 3.59 \exp 4.3, \\ \log ft(2_{gs}^- \rightarrow 3^-) &= 4.074 \exp 4.5. \end{aligned} \quad (116)$$

3.8. Calculating β^+ /EC decay strength function ft for ^{40}Sc

The β^+ / EC equation is



the calculated Q -vlaue is

$$Q_{EC} = 14.320 \text{ MeV}. \quad (118)$$

According to the SP states of ^{40}Ca shown in fig.(10) and using eq.(92) initial nuclear state,

$$|\Psi_i\rangle = |p_i n_i^{-1}\rangle = \left| \pi 0f_{\frac{7}{2}}(\nu 0d_{\frac{3}{2}})^{-1} \right\rangle, \quad (119)$$

whereas the final nuclear state

$$|\Psi_f\rangle = c_\pi^\dagger h_\pi^\dagger |HF\rangle = |p_f p_f^{-1}\rangle \text{ or } \Psi_f = c_\nu^\dagger h_\nu^\dagger |HF\rangle = |n_f n_f^{-1}\rangle. \quad (120)$$

We have the following possible SP transitions:

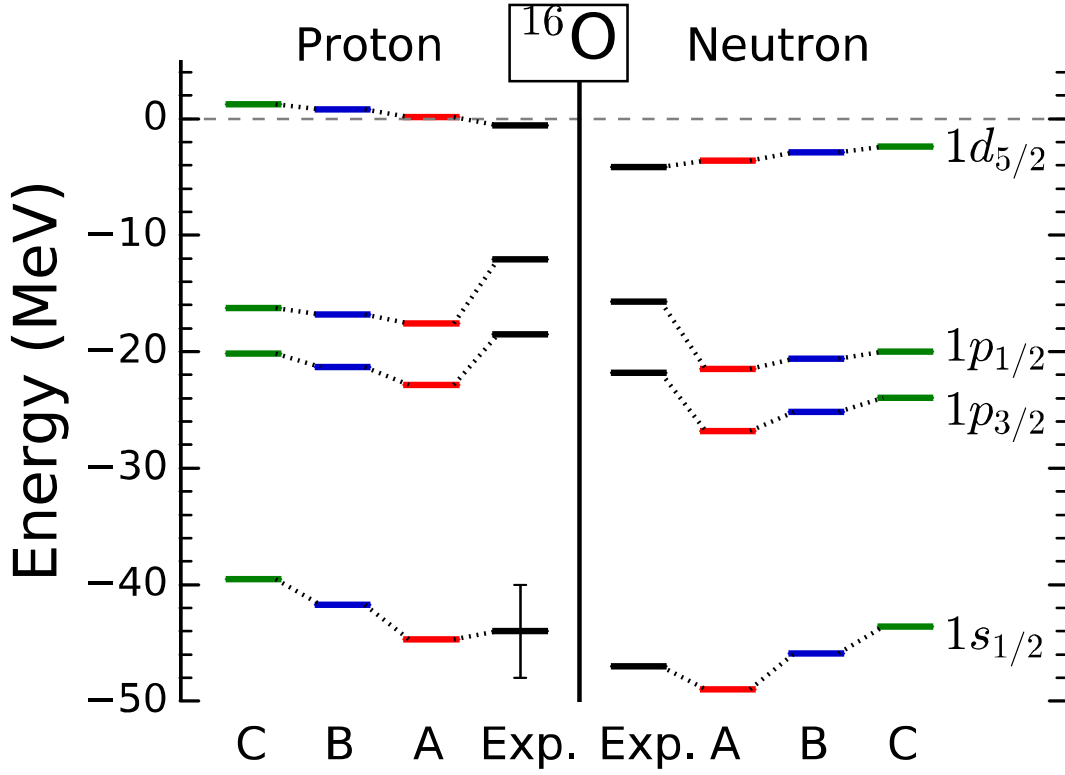


Figure 9: Single-particle energies of ^{16}O calculated by the relativistic Brueckner-Hartree-Fock (RBHF) theory using the interactions Bonn A, B, and C [28], in comparison with experimental data in [29].

1- $\pi 0d_{\frac{3}{2}}(-6.2 \text{ MeV}) \rightarrow \nu 0d_{\frac{3}{2}}(-14.2 \text{ MeV}) \Rightarrow \pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{3}{2}})^{-1}$: Thus $\Delta E = 8 \text{ MeV}$ The GT decay amplitude using eq.(96) becomes

$$\begin{aligned}
\left\langle \pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{3}{2}})^{-1}; J_f \parallel \beta_L^+ \parallel \pi 0f_{\frac{7}{2}}(\nu 0d_{\frac{3}{2}})^{-1}; 4 \right\rangle &= (-1)^{\frac{7}{2} + \frac{3}{2} + J_f + 1} \times \\
\sqrt{2 \times 4 + 1} \hat{J}_f \sqrt{2 \times 1 + 1} \left\{ \begin{matrix} 4 & J_f & 1 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\} \mathcal{M}_L(0d_{\frac{3}{2}} \rightarrow 0d_{\frac{3}{2}}) & \quad (121) \\
&= (-1)^{J_f} \sqrt{9} \sqrt{3} J_f \times \\
&\quad \left\{ \begin{matrix} 4 & J_f & 1 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\} \left(-\frac{2}{\sqrt{5}} \right) \\
&= (-1)^{J_f} \left(-6 \sqrt{\frac{3}{5}} \right) \hat{J}_f \left\{ \begin{matrix} 4 & J_f & 1 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\} \\
&= A_1(J_f).
\end{aligned}$$

Only $J_f = \{3, 4, 5\}$ are allowed. $A_1(3) = -1.54919$, $A_1(4) = -1.07331$, and $A_1(5) = 1.3594$. Make use of eq.(121) into eq.(9), we obtain

$$\begin{aligned}
B_{GT}(4^- \rightarrow 3) &= \frac{g_A^2}{2J_i + 1} | \langle ^-4 | B_{GT} | 3 \rangle |^2 \\
&= \frac{(1.25)^2}{2 \times 4 + 1} (-1.54919)^2 = 0.416665 \quad (122)
\end{aligned}$$

$$B_{GT}(4^- \rightarrow 4) = \frac{g_A^2}{2J_i + 1} | \langle ^-4 | B_{GT} | 4 \rangle |^2 = 0.199999 \quad (123)$$

For the Fermi transition, the Fermi decay amplitude becomes,

$$\begin{aligned}
\left\langle \pi 0 f_{\frac{7}{2}} (\pi 0 d_{\frac{3}{2}})^{-1}; J_f \parallel \beta_L^+ \parallel \pi 0 f_{\frac{7}{2}} (\nu 0 d_{\frac{3}{2}})^{-1}; 4 \right\rangle &= (-1)^{\frac{7}{2} + \frac{3}{2} + J_f + 1} \times \\
\sqrt{2 \times 4 + 1} \hat{J}_f \sqrt{2 \times 0 + 1} \left\{ \begin{matrix} 4 & J_f & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\} \mathcal{M}_0(0 d_{\frac{3}{2}}) &\rightarrow 0 d_{\frac{3}{2}} \\
&= 9(-1)^4 \left\{ \begin{matrix} 4 & J_f & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\} (2) \\
&= -3.
\end{aligned} \tag{124}$$

Only $J_f = 4$ contributes to the amplitude (124)

$$\langle 4^- | \beta_F | 4 \rangle = -3.$$

Using eq.(124) into eq.(8)

$$B_F = \frac{(1.0)^2}{2 \times 4 + 1} (-3)^2 = 1. \tag{125}$$

Make use of eq.(121) into eq.(9), we obtain

$$B_{GT}(4^- \rightarrow 5) = \frac{g_A^2}{2J_i + 1} |\langle 4^- | \beta_{GT} | 5 \rangle|^2 = 0.320828. \tag{126}$$

Make use eq.(122) and eq.(123) and eq.(125) and eq. (126) into eq.(6) ,and eq.(11), we obtain, $\log ft(4^- \rightarrow 3) = 4.16$ experimental value is 4.8

$\log ft(4^- \rightarrow 4) = 3.70$ experimental value is 4.6

$\log ft(4^- \rightarrow 5) = 4.28$ experimental value is 4.7.

2- $\pi 0 d_{\frac{5}{2}}(-12.0 \text{ MeV}) \rightarrow \nu 0 d_{\frac{3}{2}}(-14.2 \text{ MeV}) \Rightarrow \pi 0 f_{\frac{7}{2}} (\pi 0 d_{\frac{5}{2}})^{-1}$: Thus $\Delta E = 2.2 \text{ MeV}$ Using eq.(96), the Gamow-Teller decay amplitude becomes

$$\begin{aligned}
\left\langle \pi 0 f_{\frac{7}{2}} (\pi 0 d_{\frac{5}{2}})^{-1}; J_f \parallel \beta_L^+ \parallel \pi 0 f_{\frac{7}{2}} (\nu 0 d_{\frac{3}{2}})^{-1}; 4 \right\rangle &= \\
(1)^{J_f + 1} \sqrt{2 \times 4 + 1} \hat{J}_f \sqrt{2 \times 1 + 1} &\left\{ \begin{matrix} 4 & J_f & 1 \\ \frac{5}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\} \frac{4}{\sqrt{5}}, \\
&= (1)^{J_f + 1} (-12 \sqrt{\frac{3}{5}}) J_f \left\{ \begin{matrix} 4 & J_f & 1 \\ \frac{5}{2} & \frac{3}{2} & \frac{7}{2} \end{matrix} \right\}, \\
&= A_2(J_f)
\end{aligned} \tag{127}$$

only $J_f = \{3, 4, 5\}$, are allowed $A_2(3) = 1.34164$ and $A_2(4) = 2.66983$ and $A_2(5) = 3.55978$. Use eq.(127) and eq.(9) we get

$$B_{GT}(4^- \rightarrow 3) = \frac{g_A^2}{2J_i + 1} |\langle 4^- | B_{GT} | 3 \rangle|^2 = \frac{(1.25)^2}{2 \times 4 + 1} (1.34164)^2 = 0.3125, \tag{128}$$

$$B_{GT}(4^- \rightarrow 4) = \frac{g_A^2}{2J_i + 1} |\langle 4^- | B_{GT} | 4 \rangle|^2 = \frac{(1.25)^2}{2 \times 4 + 1} (2.66983)^2 = 1.2375, \tag{129}$$

$$B_{GT}(4^- \rightarrow 5) = \frac{g_A^2}{2J_i + 1} |\langle 4^- | B_{GT} | 5 \rangle|^2 = \frac{(1.25)^2}{2 \times 4 + 1} (3.55978)^2 = 2.20001. \tag{130}$$

The Fermi transition is zero for this transition for any value of J_f . Use eq.(128), eq.(129), and eq. (130) into eq.(6) and eq.(11) , we obtain,

$\log ft(4^- \rightarrow 3) = 4.29$ experimental value 5.1

$\log ft(4^- \rightarrow 4) = 3.69$ experimental value 3.3

$\log ft(4^- \rightarrow 5) = 3.44$ experimental value 4.7 (doesn't exist experimentally for such energy range).

Complete results are shown in table (6).

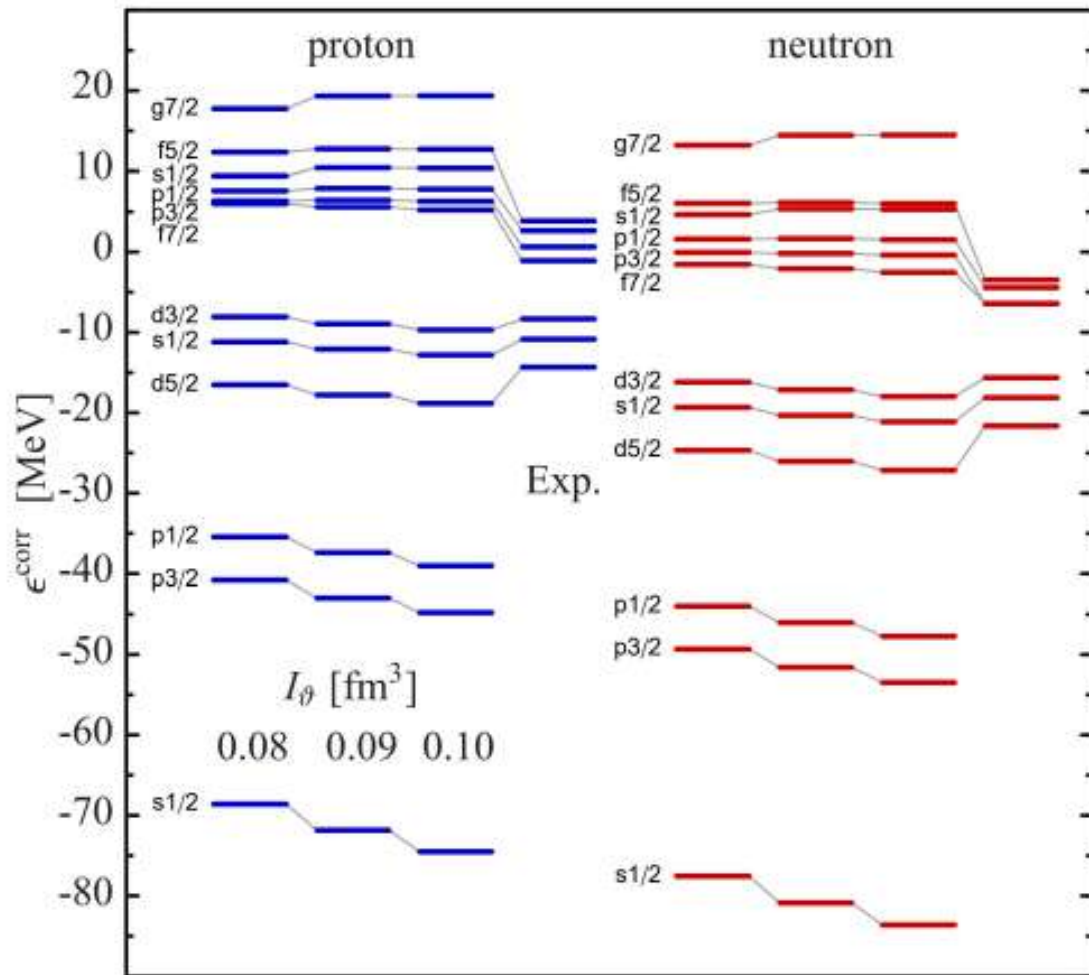


Figure 10: Single-particle energies of ^{40}Ca calculated by Hartree-Fock (RBHF) theory using the interactions AV18 [30], in comparison with experimental data from [31].

Table 6: The calculated strength function $\log ft$ compared with the experimental values for the ^{40}Sc to ^{40}Ca β^+ /EC.

Transition	$\log ft$	energy (MeV)	Isospin T	$\log(ft)_{\text{exp}}$
$ \langle 4^- \beta_{GT} 3 \rangle $	4.16	3.73	0	4.8
$ \langle 4^- \beta_{GT} 4 \rangle $	3.70	5.61	0	4.6
$ \langle 4^- \beta_{GT} 5 \rangle $	4.28	4.49	0	4.7
$ \langle 4^- \beta_{GT} 3 \rangle $	4.29	6.58		5.1
$ \langle 4^- \beta_{GT} 4 \rangle $	3.69	7.65	1	3.3
$ \langle 4^- \beta_{GT} 5 \rangle $	4.44	4.49	0	4.7

4. Conclusion

We summarize the results of $\log(ft)$ calculated using particle-hole theory in table (7) for ^{16}Ni decay and table (8) for ^{40}Sc EC.

Table 7: Summary of the calculated β^- -decay logarithm of the strength function $\log ft$ for all allowed transitions, using particle-hole theory for the decay $^{16}\text{N} \rightarrow ^{16}\text{O} + e^- + \bar{\nu}_e$ which has $Q_\beta = 10.419$ MeV. The temperature calculation is based on the hot Thomas-Fermi model [32].

Single Particle Transition	SP transition (MeV)	Nuclear transition	Nuclear state (MeV)	Isospin T	$\log ft$ Theory	$\log ft$ exp	Temperature (MeV) [32]	$g_A(\text{exp})$
$\nu 0p_{\frac{1}{2}} \rightarrow \pi 0p_{\frac{1}{2}}$ $\nu 0d_{\frac{5}{2}} (\nu 0p_{\frac{1}{2}})^{-1}$	-4	$\langle 2^- \beta_{GT} 2^- \rangle$	8.87	0	3.59	4.3	1.8347	0.55464
$\nu 0d_{\frac{5}{2}} \rightarrow \pi 0d_{\frac{5}{2}}$ $\pi 0d_{\frac{5}{2}} (\pi 0p_{\frac{1}{2}})^{-1}$	-3.5	$\langle 3^- \beta_{GT} 2^- \rangle$	6.13	0	4.074	4.5	1.5135	0.76535

We utilize the one-particle-hole theory to calculate the half-lives of odd-even nuclei of ^{15}O , ^{17}F , ^{39}Ca , and ^{41}Sc . By comparing the calculated values with the experimental data we found discrepancies ranging from 27.4% for ^{39}Ca to 41.4% for ^{17}F . The experimental half-lives always exceed those of the theoretical values. This is attributed to the fact that the one-particle-hole theory is dependent on the SP states and doesn't account for the residual force among the even valence nucleons. This suppresses the SP transitions in the initial and final nuclear wave functions Ψ_i and Ψ_f . The simplicity of this model makes it a favorable technique to obtain a rough estimate for the half-lives of the β -decays

In the two-particle-hole scheme, we calculate the strength function $\log ft$ of the β^+ /EC decay of the even-even nucleus ^{56}Ni . The value is $\log ft = 2.46$. The experimental value is 4.4 which exceeds the calculated value $\log ft = 4.4$. A 44.1% discrepancy is an indication of the failure of the particle-hole theory. Again, as shown in table (4), the transition is unfavoured allowed, meaning that the single-particle transition is suppressed during the initial and the final states due to the residual two-body interaction.

For odd-odd nuclei, ^{16}N and ^{40}Sc , the results take different turn. Except for $\pi 0d_{\frac{5}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ transition, corresponds to $\langle 4^- | \beta_{GT} | 4^- \rangle$ amplitude, for the ^{40}Sc , all experimental values of the strength functions are less than those of the experimental ones. The effect of the NN residual force is diluted for the odd-odd nuclei. The discrepancies here are merely due to the error in determining the exact energy levels of the SP states. The theory is a success in this situation.

Table 8: Summary of the calculated β^+/EC -decay logarithm of the strength function $\log ft$ for all allowed transitions, using particle-hole theory for the decay $^{40}\text{Sc}+e^- \rightarrow ^{40}\text{Ca}+\nu_e$ which has $Q_{EC} = 14.32$ MeV. The temperature calculation is based on the hot Thomas-Fermi model [32].

Single Particle Transition	SP transition (MeV)	Nuclear transition	Nuclear state (MeV)	Isospin T	$\log ft$ Theory	$\log ft$ exp	Temperature (MeV) [32]	$g_A(\text{exp})$
$\pi 0d_{\frac{3}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ $\pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{3}{2}})^{-1}$	8	$\langle 3^- \beta_{GT} 4^- \rangle$	3.73	0	4.16	4.8	0.847	0.604433
$\pi 0d_{\frac{5}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ $\pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{5}{2}})^{-1}$	2.2	$\langle 3^- \beta_{GT} 4^- \rangle$	6.58	?	4.29	5.1	1.090	0.494102
$\pi 0d_{\frac{3}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ $\pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{3}{2}})^{-1}$	8	$\langle 4^- \beta_{GT} 4^- \rangle$	5.61	0	3.70	4.6	1.013	1.09832
$\pi 0d_{\frac{5}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ $\pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{5}{2}})^{-1}$	2.2	$\langle 4^- \beta_{GT} 4^- \rangle$	7.65	1	3.69	3.3	1.169	1.97228
$\pi 0d_{\frac{3}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ $\pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{3}{2}})^{-1}$	8	$\langle 5^- \beta_{GT} 4^- \rangle$	4.49	0	4.28	4.7	0.918	0.772868
$\pi 0d_{\frac{5}{2}} \rightarrow \nu 0d_{\frac{3}{2}}$ $\pi 0f_{\frac{7}{2}}(\pi 0d_{\frac{5}{2}})^{-1}$	2.2	$\langle 5^- \beta_{GT} 4^- \rangle$	4.49	0	4.44	4.7	0.918	0.295141

The temperature-dependent of the $\log ft$ is shown in fig.(11). The general trend is that the value of $\log ft$ is slowly decreasing with temperature. The fluctuations in the values reflect the dependent on the shell configurations used to compute the amplitude. The amplitude $\langle 3^-|\beta_{GT}|2^- \rangle$ for ^{16}N and $\langle 5^-|\beta_{GT}|4^- \rangle$ for ^{40}Sc have the closer values to the experimental ones. The drop of the $\log ft$ value for $\langle 4^-|\beta_{GT}|4^- \rangle$ amplitude is attributed to the fact that Gamow-Teller transition is more likely to occur for $\Delta J = 0$.

We need to go more steps further and use two-particle-two-hole configurations to “quench” the Gamow-Teller strengths [33] and implement quasi-particle random phase approximation [34] for more accurate computations.

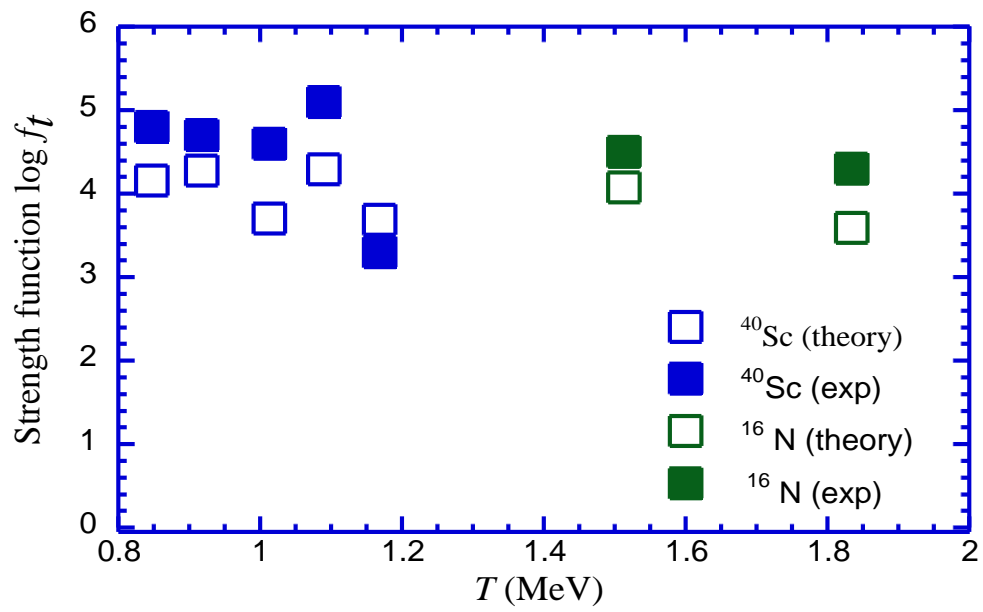


Figure 11: The temperature-dependent of the theoretical and experimental strength function $\log f_t$ for ^{40}Sc and ^{16}N (see colored legends).

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