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Time-triggered Impulsive Control for Fractional-order Chaotic Financial System

Mohsen DLALA [1], Farida ALHARBI [1]

[1]Department of Mathematics, College of Science, Qassim University, 51452, Buraydah, Saudi Arabia

Abstract

The exponential stabilization of fractional-order differential systems is examined through the utilization of time-triggered impulsive control (TTC). The impulsive control strategy is characterized by a triggering mechanism, which relies on the states of the systems. Updates to the controller exclusively occur at impulsive moments. Decreasing the frequency of controller updates leads to a reduction in the utilization of communication bandwidth and computational resources. Moreover, the effectiveness of the theoretical findings is demonstrated through a numerical illustration involving simulation on a chaotic fractional order system within the domain of finance.

1 Introduction

Fractional order systems emerges as a versatile mathematical framework with profound implications across numerous domains. Its ability to capture non local, memory-dependent [1], and anomalous behaviors provides researchers and practitioners with powerful tools to tackle complex real-world problems such as finance [2, 3, 4, 5], engineering [6], and biological systems [7]. At the same time, Control theory plays a critical role in managing complex dynamic systems across various fields, including engineering, biology, and economics. Two primary approaches in control theory are continuous control and impulsive control. Continuous control involves smooth, ongoing adjustments to a system's parameters to maintain desired behaviors, typically modeled through differential equations. While this method is well-established and effective in many applications, it requires constant monitoring and adjustment, which can be computationally expensive and practically challenging for real-time systems [8, 9, 10, 11].

In contrast, impulsive control involves sudden, discrete changes to the system's state at specific moments in time. These impulses can be strategically timed to achieve desired outcomes with fewer interventions, making impulsive control particularly appealing in scenario where continuous monitoring is impractical or where sudden, decisive actions are more effective. Impulsive control has been successfully applied across various domains, including biological systems, spacecraft trajectory correction, and economic systems [12, 13, 14].

In recent years, there has been growing interest in applying control theory to fractional-order chaotic financial systems, which are characterized by their non-linearity and memory effects. These systems are highly sensitive to initial conditions, making them difficult to control using traditional continuous methods [15]. Fractional calculus, which extends the concept of integer-order derivatives to non-integer orders, offers a powerful framework for modeling these complex systems [16, 17, 1, 18]. The use of fractional-order models provide a more accurate representation of the dynamics within financial systems by capturing long-range dependencies and complex temporal structures that cannot be fully addressed using integer-order models [19, 20, 21, 22].

Given the chaotic nature of these systems, impulsive control offers distinct advantages over continuous control. The ability to apply targeted, high-impact interventions at critical moments can help stabilize the system and prevent undesirable outcomes, such as financial crises. Moreover, impulsive control is more efficient in scenarios where continuous monitoring is either too costly or unfeasible, making it an attractive option for managing the inherent volatility of financial markets [20, 21, 22].

In [20], a system of differential equations was proposed to model the interactions among various factors in a financial system. The system exhibited unpredictable and chaotic behaviors, highly sensitive to initial conditions, with the time history of the system displaying pseudo-random behaviors indicative of chaotic dynamics. The study also identified complex dynamical behaviors such as period-doubling bifurcations, bifurcation diagrams demonstrated that chaotic behaviour occurred across a wide range of system parameters. These findings suggest that the interactions among key factors in the financial model contribute to the emergence of such dynamics.

In [21], an investigation has been conducted on the dynamics of a financial system incorporating fractional order and robust chaotic control, employing both analytic and numerical techniques. A control strategy based on robust fractional-order sliding mode control has been formulated utilizing Lyapunov stability theory to achieve stability of chaotic trajectories.

In [22] a synchronization criterion for fractional-order hyper-chaotic financial systems using impulsive control and state feedback controllers was presented. The authors established a global Mittag-Leffler synchronization criterion that allowed a backward economic system to synchronize asymptotically with an advanced economic system through effective macroeconomic management.

Motivated by the above discussions, this chapter focuses on the problem of exponential stability for nonlinear fractional-order systems via impulsive control strategy and its application to chaotic financial systems. Impulsive control in finance can manifest through mechanisms such as government interventions during economic crises, central banks adjusting interest rates, sudden market fluctuations due to external shocks, regulatory changes affecting financial markets, and efforts to stabilize exchange rates during currency crises. The rest of the paper is organized as follows. In Section 2, the fractional tools are reviewed. In Section 3, the problem is formulated, the fractional financial chaotic model is explained, and the main result is proved. In Section 4, The exponential stability of chaotic financial systems is proved, along with a numerical simulation. The paper concludes with a conclusion and future work in Section 5.

2 Mathematical preliminaries

In this section, we review key definitions from stability theory and introduce a chaotic financial system in the context of impulsive control. In this section, we introduce some definitions of fractional calculus, recall some important results we'll be using later, and introduce our chaotic system to finance.

2.1 Caputo fractional-order derivative

For $\alpha \in (0,1)$ and $T \geq t_0 \geq 0$, the Caputo fractional derivatives is defined as follows

$$ {}^{c}D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-\tau)^{-\alpha} f'(\tau) d\tau \ t_0 < t \leq T. \tag{1}$$

where $\Gamma(s) = \displaystyle\int_{t_0}^{\infty} t^{s-1} e^{-t} dt, s > 0$ denotes the Gamma function and $f'(\tau)$ denotes the derivative of f at point τ. The one-parameter Mittag-Leffler function is an important tool in fractional calculus and is defined as follows

Definition 1. *[23] For any complex number z and $\alpha > 0$, we define the complex function*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2}$$

The function E_{α} is known as the one-parameter Mittag-Leffler function.

The following proposition gives an important proprite of the one-parameter Mittag-Leffler function E_{α}.

Proposition 1. *[24] Let $\alpha \in (0,1)$ and $t \geq t_0$. Then, for all $c > 0$ the function $t \to E_{\alpha}[c(t-t_0)^{\alpha}]$ is non-negative and it is monotonically non-decreasing and $E_{\alpha}[c(t-t_0)^{\alpha}] \geq 1$.*

Let $\alpha \in (0,1)$. Consider the Caputo fractional non-linear system

$$ {}^{c}D_{t_0}^{\alpha} x(t) = f(t, x(t)); \quad t > t_0, \tag{3}$$
$$ x(t_0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^m$ is the system state, t_0 is the initial time, and the function $f(t,x)$ is piecewise continuous in t, locally Lipschitz in x, with constant L, and satisfies $f(t,0) = 0$ for all $t \geq t_0$.

Definition 2. *[25] An element $x_e \in \mathbb{R}^m$ is called an equilibrium point of the Caputo fractional dynamic system (3)-(4) if $f(t, x_e) = 0, \forall t \geq t_0$.*

Definition 3. *[25] The origin $x = 0$ of the system (3)-(4) is exponentially stable if there exist positive constants c, γ and δ such that*

$$\|x(t)\| \leq c \|x(t_0)\| \, e^{-\gamma(t-t_0)}, \tag{5}$$

for all $x_0 \in B_{\delta} := \{x \in \mathbb{R}^m, \|x\| < \delta\}$ and $t \geq t_0$.

Proposition 2. *[25] Let $\alpha \in (0,1)$, and $x = 0$ an equilibrium point of the system (3)-(4). If the function f is Lipschitz continuous with respect to x with Lipschitz constant L and piecewise continuous with respect to t, then the solution of system (3)-(4) satisfies*

$$\|x(t)\| \leq \|x_0\| \ E_{\alpha}\big(L(t-t_0)^{\alpha}\big). \tag{6}$$

2.2 Fractional Chaotic System in Finance

Fractional-order derivatives are characterized by their inclusion of a memory component, allowing them to account for previous states the system's analysis. This feature enhances their effectiveness and accuracy compared to integer-order derivatives when representing phenomena influenced by historical data, such as genetic diseases, economics systems, and financial markets. It is deemed to be the most effective to study the financial system using a fractional differential equation model, which incorporates interest rates, investment demand, and price indexes, since it considers the past values of the variables, which helps clarify how the present financial situation is influenced by past events. In [21] the following model is proposed:

$$
\begin{aligned}
{}^cD_0^{q_1}x_1(t) &= gx_3 + x_1x_2 - ax_1, &(7)\\
{}^cD_0^{q_2}x_2(t) &= -bx_2^2 - sx_1^2 + r, &(8)\\
{}^cD_0^{q_3}x_3(t) &= -cx_3 + -\beta x_1 - px_2, &(9)
\end{aligned}
$$

where a is the saving rate, b is the investment cost, and c is the elasticity of demand. The system (7)-(9) is called a commensurate fractional-order system if $q_1 = q_2 = q_3 = q$; otherwise, it is considered an incommensurate fractional-order system [26]. The system (7)-(9) exhibits chaotic behavior when $q = 0.83, a = 0.3, b = 0.04, c = 1, r = 1, s = 0.1, p = 0, g = 1.2, \beta = 1$, and initial conditions $(1.2, 1.5, 1.6)$ are considered (see Figures 1 and 2). Let $x = (x_1, x_2, x_3)^T$, the system (7)-(9) takes the form of system (3)-(4) with $f(t, x) = (gx_3 + x_1x_2 - ax_1, r - bx_2^2 - sx_1^2, -cx_3 - \beta x_1 - px_2)^T$.

Our next step is to develop a time-triggered impulsive control strategy that guarantees the exponential convergence of the system (7)-(9).

3 Main result

In this section, we present our main result: the construction of a sequence $(t_k, u(x(t_k)))$, where moments t_k are carefully selected to influence the state by implementing the value $u(x(t_k))$ in the state at time t_k, resulting in an exponential convergence to zero for system (3)-(4) under control.

Let $N \in \mathbb{N}, \delta$ positive constant, and $0 < \lambda < 1$. Consider the following set

$$
\mathcal{I}_k = \left\{ i, 1 \leq i \leq N, \|x(t_k + i\delta)\| \geq \lambda \|x(t_k^+\|\right\}, \tag{10}
$$

and put $i_k = \min \mathcal{I}_k$.

The following time-triggered mechanism defines the condition for intervening at time t_{k+1} to control the system if t_k has already occurred:

$$
\mathcal{E} : \begin{cases} t_{k+1} = t_k + i_k\delta, \quad x(t_{k+1}^+) = x(t_{k+1}) + u((x(t_{k+1})), \text{ if } \mathcal{I}_k \neq \emptyset, \\ t_{k+1} = t_k + N\delta, \quad x(t_{k+1}^+) = x(t_{k+1}), \text{ if } \mathcal{I}_k = \emptyset. \end{cases}
$$

Now, we can establish our main result concerning the exponential stability of the following control system:

$$
\begin{aligned}
{}^cD_{t_0}^{\alpha}x(t) &= f(t, x(t)); \quad t \neq t_k, \quad k = 1, 2, ... &(11)\\
x(t_k^+) &= x(t_k) + u(x(t_k)); &(12)\\
x(t_0^+) &= x_0. &(13)
\end{aligned}
$$

Theorem 1. *Assume that there exists a positive constant $R > 0$, and $\mu \in (0, 1)$ such that for any $x \in B_R$,*

$$\|x + u(x)\| \leq \mu\|x\|. \tag{14}$$

Then, the origin $x = 0$ of the system (11)-(13), where the sequence $(t_k, u(x(t_k))$ is defined by mechanism $\mathcal{E}$, is exponentially stable.

Proof. According to mechanism $\mathcal{E}$, $x(t_k)$ is defined depending on whether t_k comes from the case $\mathcal{I}_k = \emptyset$ or from the case $\mathcal{I}_k \neq \emptyset$.

When $\mathcal{I}_k = \emptyset$, we have:

$$\begin{aligned} \|x(t_{k+1}^+)|\| &= \|x(t_{k+1})\| \\ &\leq \lambda\|x(t_k^+)\|. \end{aligned} \tag{15}$$

if $\mathcal{I}_k \neq \emptyset$, we have:

$$\begin{aligned} \|x(t_{k+1}^+)\| &= \|x(t_{k+1}) + u(x(t_{k+1}))\| \\ &\leq \mu\|x(t_{k+1})\| \\ &\leq \mu\|x((t_k)^+)\|E_\alpha(L(t_{k+1} - t_k)^\alpha) \\ &\leq \mu E_\alpha(L(N\delta)^\alpha)\|x((t_k)^+)\|. \end{aligned} \tag{16}$$

Combining (15), and (16) we obtain for all $k \geq 0$:

$$\|x(t_{k+1}^+)|\| \leq \beta\|x((t_k)^+)\|, \tag{17}$$

where $\beta = \max\{\lambda, \mu E_\alpha(L(N\delta)^\alpha)\}$.

By induction, we can conclude that for all $k \geq 1$:

$$\|x(t_k^+)\| \leq \beta^k\|x(t_0^+)\|. \tag{18}$$

Now, using Proposition 1, inequalities (6) and (18), we obtain for all $t \in (t_k, t_{k+1}]$:

$$\begin{aligned} \|x(t)\| &\leq E_\alpha(L(N\delta)^\alpha)\beta^k\|x(t_0^+)\| \\ &\leq E_\alpha(L(N\delta)^\alpha)e^{k\ln(\beta)}\|x(t_0^+)\|. \end{aligned} \tag{19}$$

Furthermore, the construction of the sequence (t_k) leads to the conclusion that for $t \in (t_k, t_{k+1}]$ we have:

$$0 < t - t_0 \leq (k + 1)N\delta. \tag{20}$$

this gives:

$$k \geq \frac{t - t_0}{N\delta} - 1. \tag{21}$$

This combined with (19), give us:

$$\begin{aligned} \|x(t)\| &\leq E_\alpha(L(N\delta)^\alpha)e^{\frac{\ln(\beta)}{N\delta}(t-t_0)}e^{-\ln(\beta)}\|x(t_0^+)\| \\ &\leq c\ \|x(t_0^+)\|\ e^{-\gamma(t-t_0)}, \end{aligned} \tag{22}$$

where $c = \dfrac{1}{\beta}E_\alpha(L(N\delta)^\alpha)$ and $\gamma = -\dfrac{\ln(\beta)}{N\delta}$. This proves the exponential stability of system (11)-(13) and achieves the proof of Theorem 1. □

4 Exponential Stabilization of Fractional-Order Chaotic Financial Systems

In this section, we will apply the results developed in Section 2 to stabilize the system (7)-(9). Since $x = 0$ is not an equilibrium point for the system (7)-(9), but $x_e = (0, 5, 0)^T$ is, we will first put $y = x - x_e$ to align with the framework of Theorem 1. The system (7)-(9) can then expressed in the following form:

$$
\begin{aligned}
{}^cD_{t_0}^{\alpha} y(t) &= f(t, y(t)); \quad t > t_0 && (23)\\
y(t_0^+) &= y_0. && (24)
\end{aligned}
$$

Where, $f(t, y) = \big(gy_3 + (y_2 + 5 - a)y_1, b(y_2 + 5)^2 - sy_1^2 + r, -cy_3 + -\beta y_1 - p(y_2 + 5)\big)^T$. Consider the control u defined by

$$
u(y(t)) = C(t)y(t), \tag{25}
$$

where, the matrix $C(t)$ is given by

$$
C(t) = \begin{pmatrix} -1 & 0 & -\frac{4}{5}\cos(3t) \\ 0 & -1 - \frac{4}{5}\sin(2t) & 0 \\ -\frac{4}{5}\sin(3t) & 0 & -1 \end{pmatrix}. \tag{26}
$$

The controlled system (23)-(24) takes the following form:

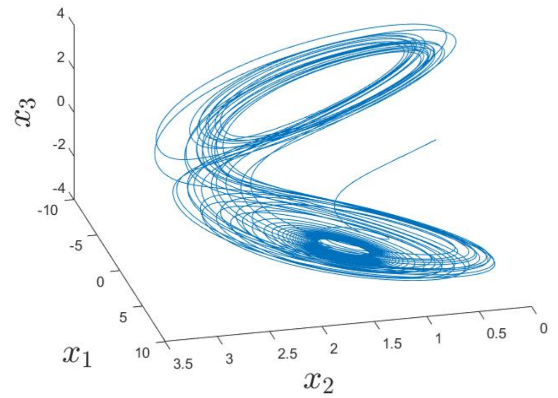
$$
\begin{aligned}
{}^cD_{t_0}^{\alpha} y(t) &= f(t, y(t)), \quad t \neq t_k; k = 1, 2, ... && (27)\\
y(t_k^+) &= \begin{cases} y(t_k) + Cy(t_k), & \mathcal{I}_k \neq \emptyset, \\ y(t_k), & \mathcal{I}_k = \emptyset. \end{cases} && (28)
\end{aligned}
$$

Since the components of f are polynomial, f is a continuously differentiable function and therefore locally Lipschizian with respect to the variable y. Moreover, it can be verified that for all $y \in \mathbb{R}^3$, inequality (14) is satisfied with $\mu = 4/5$. Thus all assumptions of Theorem 1 are verified, and the system (27)-(28) is exponentially stable.

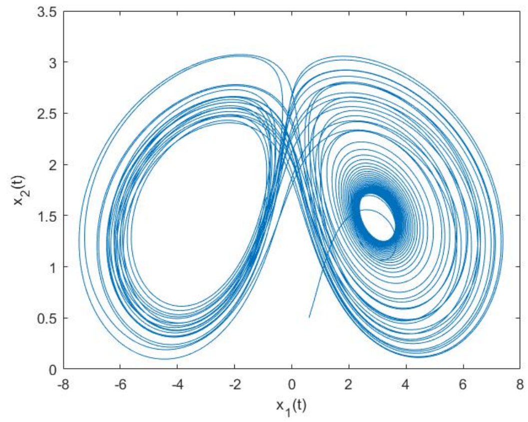
The fractional differential systems (7)-(9) and (27)-(28) are simulated with MATLAB R2017a. The methodology adopted for the numerical solution of these systems involves the use of the `fde12` solver specifically designed for the treatment of fractional differential equations, a model introduced in [27] and subsequently updated in the implementation detailed in [28]. The use of this solver underlines the commitment to precision and accuracy in the computational treatment of these complex systems and marks a significant advance in the field of fractional calculus and its practical applications.

The state trajectories of the fractional order system (7)-(9) are depicted in Figure 1 and Figure 2. The chaotic nature of the uncontrolled system is clearly evident.

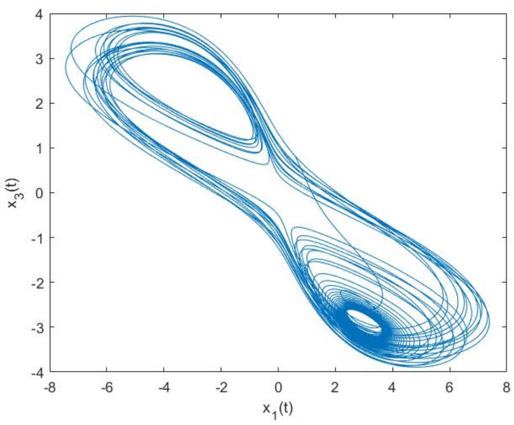
Figure 3 illustrates the outcomes of the numerical simulation, demonstrating the exponential convergence of solutions of system (27)–(28) under the time-triggered control described by the mechanism $\mathcal{E}$ (3a). Sub-figure 3a shows the convergence to zero of the y_i components of system (27)–(28), sub-figure 3b illustrates the exponential convergence to zero of the solution parameter y(t) of system (27)–(28), while sub-figure 3c records the intervention moments to activate the control in system (27)–(28).



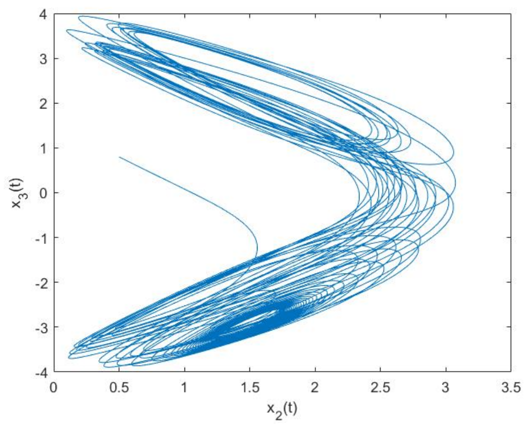
(a) Numerical simulations for $x_1 - x_2 - x_3$ phase.



(b) Numerical simulations for $x_1 - x_2$ phase.

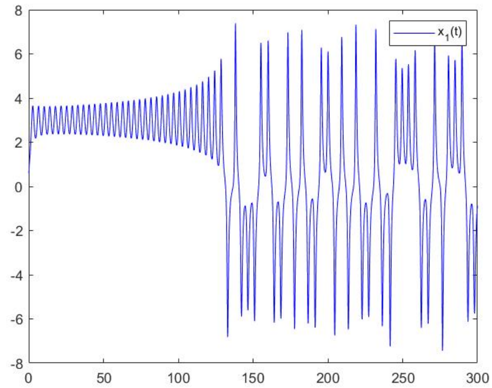


(c) Numerical simulations for $x_1 - x_3$ phase.

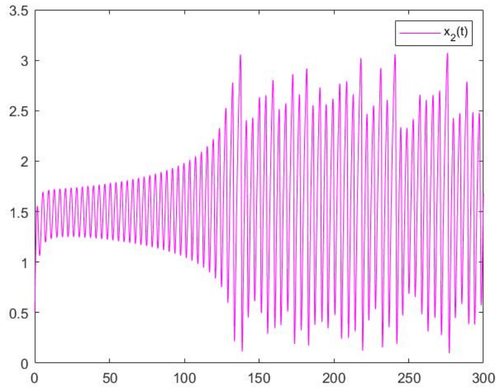


(d) Numerical simulations for $x_2 - x_3$ phase.

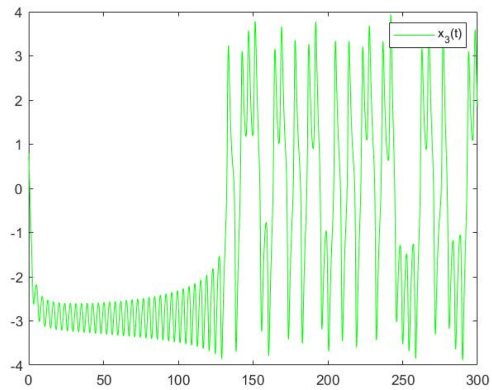
Figure 1: Chaotic behavior of system (7)–(9).



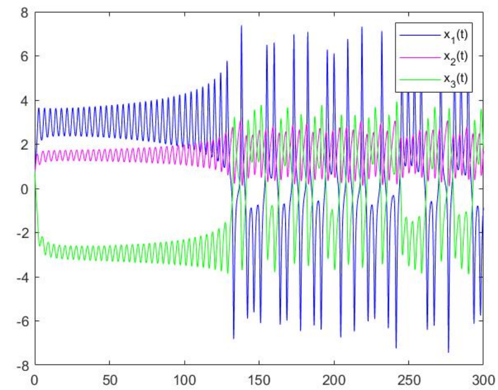
(a) Time series of x_1.



(b) Time series of x_2.

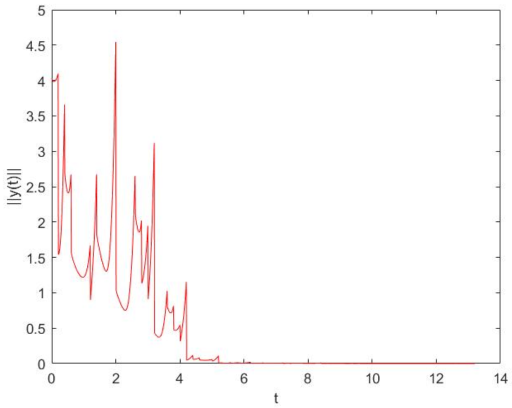


(c) Time series of x_3.

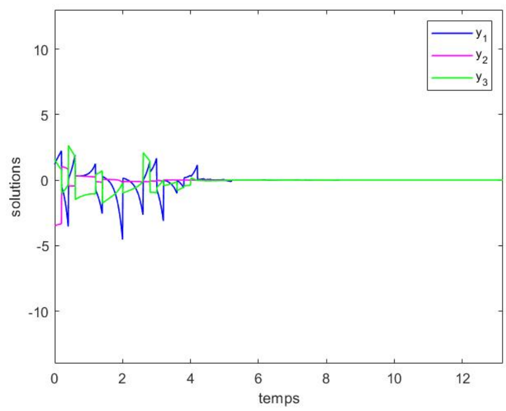


(d) Time series of the solution x.

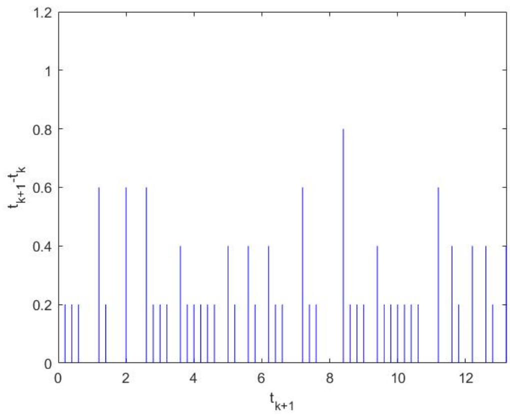
Figure 2: Chaotic behaviours of the solution x(t) of the system (7)–(9).



(a) Convergence of $\|y(t)\|$.



(b) Convergence of $y_i(t)$.



(c) Impulsive instants.

Figure 3: System (27)–(28) under TTM ($\mathcal{E}$).

5 Conclusion

In this paper, rapid exponential stabilization via time-triggered control has been investigated for fractional-order systems. As an application of the obtained theoretical results, the exponential stabilization of fractional-order chaotic systems in finance is also presented in the simulation example. The exponential stabilization of fractional-order dynamical systems via event-driven impulsive control with delay will be investigated in future studies.

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