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Analysis of a resource-consumer mathematical model

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Abstract: The research investigates the dynamics of a resource-consumer mathematical model. The model, which consists of differential equations, was previously published by [9]. In this paper, we analyze the existence of solutions, provide the numerical solutions, and discuss the dynamics. Furthermore, we solve the model for the steady states and confirm the existence of equilibria, which have been analyzed. The results indicate that resources and consumers can coexist if we deploy enough resources into the population; otherwise, the population collapses.

Keywords: Food and consumer differential equation model; existence of solutions; analysis of steady states

1. Introduction

Resource-consumer mathematical models are among the most popular models and have been examined by many authors theoretically and numerically [1, 2, 3, 4, 5, 6, 7, 8, 9]. A predator-prey model with one resource and two consumers is considered by [6]. He et al. [5] discuss the dynamics of a consumer-resource model with diffusion. [8] study the influence of spatial memory on the solutions of a consumer-resource model. Local stability of equilibria has been discussed by [2] for the Daphnia model. Alanazi [1] derives a resource-consumer mathematical model that incorporates diffusion coefficients and home ranges to fully understand the impact of diffusion and home ranges on the dynamic behavior of the populations. Leah [4] provides an excellent work discussing mathematical models in Biology and the stability of the steady states. Thieme [9] proposes a mathematical model to explore the relations between consumers and food (the resources) on which the population lives. The mathematical model derived by [9] is

$$\frac{dV(t)}{dt} = \varphi - k\,V(t) - \frac{h}{q}\,V(t)U(t),$$
$$\frac{dU(t)}{dt} = h\,V(t)\,U(t) - w\,U(t), \qquad (1.1)$$

where $t > 0$, and k, q, w are positive. $V(t)$ is the biomass of the resources, while $U(t)$ describes the biomass of the consumers. The food deployed to the population at a constant rate φ. Also, the food degrades at a constant rate k. h is the increase in the consumer biomass per unit of time by consuming one unit of food biomass. $\frac{1}{w}$ is the life expectancy of the population U. Lastly, the amount of food biomass consumed per unit of time by per unit of consumer biomass is $\frac{h}{q}$.

The dynamics of the model (1.1) have not been fully analyzed theoretically and numerically. Therefore, this paper aims to understand the dynamics of the model described in (1.1). First, we prove the existence of solutions. Second, we show the existence of numerical solutions, which provide another helpful understanding of dynamics. In addition, we solve the model for the steady states and confirm the existence of equilibria, which have been discussed. To investigate the model for the steady-state solutions, we first find the corresponding characteristic equation for each equilibrium in the system. An equilibrium is locally asymptotically stable if all roots of the characteristic equation have negative real parts. In contrast, an equilibrium is unstable if at least one root of the characteristic equation has one positive real part. Finally, the paper provides the numerical solutions of the steady states and interprets their meaning.

Conditions and theorems of stability and linearization provided by Leah [4] are our key tools to obtain the main results. For more numerical experiments and analysis of numerical solutions, we suggest the following work [1, 9, 10, 11, 12, 13, 14].

The plan of the paper is as follows. In section 2, the model is introduced and defined along with the initial conditions. Section 3 discussed the existence of solutions. Numerical solutions of the proposed model are found in section 4. In section 5, we study the steady-state solutions of the model. Section 6 provides a numerical analysis of the steady states. In Section 7, we summarize and discuss the results.

2. A mathematical model of resources and consumers

Assume V and U are the biomass of food resources and the consumer biomass of the population, respectively. Differently than [14], we assume $h(t)$ is a function of time t. We consider the following model [14],

$$\begin{aligned} \frac{dV(t)}{dt} &= \varphi - k\,V(t) - \frac{h(t)}{q}\,V(t)U(t), \\ \frac{dU(t)}{dt} &= h(t)\,V(t)\,U(t) - w\,U(t), \end{aligned} \tag{2.1}$$

where $t > 0$, and k, q, w are positive constants. The food deployed to the population at a constant rate φ. Also, the food degrades at a constant rate k. h is the increase in the consumer biomass per unit of time by consuming one unit of food biomass. $\frac{1}{w}$ is the life expectancy of the population U. Lastly, the amount of food biomass consumed per unit of time by per unit of consumer biomass is $\frac{h}{q}$. We assume the initial conditions are given by

$$V(0) = \hat{V}, \quad U(0) = \hat{U}. \tag{2.2}$$

3. Existence of solutions

We assume that $h(0) = 0$, $\int_0^\infty h(s)\,U(s)\,ds < \infty$, and $\int_0^\infty h(s)\,V(s)\,ds < \infty$. By the first equation in (2.1), we have

$$V'^{(t)} + \left(k + \frac{h(t)}{q}\,U(t)\right)V(t) = \varphi. \tag{3.1}$$

We use the integrating factor with $\alpha(t) = e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr}$ to solve the equation (3.1). Therefore, multiplying both sides by $\alpha(t)$ we get

$$\alpha(t)V'^{(t)} + \alpha(t)\left(k + \frac{h(t)}{q}\,U(t)\right)V(t) = \alpha(t)\,\varphi. \tag{3.2}$$

This leads to

$$\begin{aligned} e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr}\,V'^{(t)} + e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr}\left(k + \frac{h(t)}{q}\,U(t)\right)V(t) \\ = e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr}\,\varphi. \end{aligned} \tag{3.3}$$

(3.3) can be reduced to

$$\frac{d}{dt}\left(e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr} V(t)\right) = e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr}\ \varphi. \tag{3.4}$$

Integrating both sides gives the following equation

$$\left(e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr} V(t)\right) = \ \varphi \int_0^t e^{kt+\frac{1}{q}\int_0^s h(r)U(r)dr}\ ds + \ \hat{V}. \tag{3.5}$$

Divide both sides by $\alpha(t) = \ e^{kt+\frac{1}{q}\int_0^t h(r)U(r)dr}$, we reach to the following expression

$$V(t) = \ \varphi \int_0^t e^{-k(t-s)-\frac{1}{q}\int_s^t h(r)U(r)dr}\ ds + \ \hat{V} e^{-kt-\frac{1}{q}\int_0^t h(r)U(r)dr}. \tag{3.6}$$

Also, by the second equation in (2.1), we have

$$U'^{(t)} + \big(w - h(t)\ V(t)\big)U(t) = \ 0. \tag{3.7}$$

We use the integrating factor with $\beta(t) = \ e^{wt-\int_0^t h(r)\ V(r)dr}$, to solve the equation (3.7). Multiplying both sides by $\beta(t)$, we get

$$\beta(t)U'^{(t)} + \beta(t)\big(w - h(t)V(t)\big)U(t) = \ 0, \tag{3.8}$$

which can be rewritten as

$$\frac{d}{dt}\big(\ \beta(t)U(t)\big) = 0. \tag{3.9}$$

Integrating (3.9) gives the following equation

$$\big(\beta(t)U(t) - \ \widehat{U}\big) = 0. \tag{3.10}$$

The latter equals

$$U(t) = \widehat{U}\ \beta^{-1}(t). \tag{3.11}$$

By (3.6) and (3.11), the solutions are bounded as $t\ \rightarrow \infty$ provided that the solutions exist.

4. Analysis of the steady-state solutions

This section uses the model in (2.1) to analyze the steady-state solutions by assuming h is a constant. The model (2.1) can be rewritten as

$$\frac{dV(t)}{dt} = \varphi - k\,V(t) - \frac{h}{q}\,V(t)U(t) =: F(V,U),$$
$$\frac{dU(t)}{dt} = h\,V(t)\,U(t) - w\,U(t) =: G(V,U), \tag{4.1}$$

where $t > 0$, and k, q, w are positive constants.

4.1 The steady states solutions

To solve for the steady state solution, we set $\frac{dV(t)}{dt} = 0$, and $\frac{dU(t)}{dt} = 0$, i.e., by solving the following system

$$\varphi - k\,V(t) - \frac{h}{q}\,V(t)U(t) = 0,$$
$$h\,V(t)\,U(t) - w\,U(t) = 0. \tag{4.3}$$

The system (4.3) has three equilibrium points. Clearly, $E_1 = (V_1, U_1) = (0, 0)$ is the first equilibrium point. The second equilibrium point is $E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$.

We can find the third equilibrium point as follows: From the second equation in (4.3), we have $V(x,t) = \frac{w}{h}$. Therefore, the first equation in (4.3) provides $U(x,t) = \frac{q\varphi}{w} + \frac{kq}{h}$. We conclude that the third equilibrium point is $E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)$.

4.2 The stability of the steady states

The Jacobian of the model (4.1) is

$$J_i = \begin{bmatrix} \frac{dF}{dV_i} & \frac{dF}{dU_i} \\ \frac{dG}{dV_i} & \frac{dG}{dU_i} \end{bmatrix}, \tag{4.5}$$

where $i = 1,2,3$. Therefore, we have

$$J_i = \begin{bmatrix} -k - \frac{h}{q}\,U_i & -\frac{h}{q}\,V_i \\ h\,U_i & h\,V_i - w \end{bmatrix}, \tag{4.5}$$

where $i = 1,2,3$. We can find the corresponding characteristic equation by calculating $\det\,(J - \lambda\,I)$, where det is the determinant and I is the corresponding identity matrix. So,

$$(J - \lambda\,I) = \begin{bmatrix} -k - \frac{h}{q}\,U - \lambda & -\frac{h}{q}\,V \\ h\,U & h\,V - w - \lambda \end{bmatrix}, \tag{4.6}$$

and

$$det(J - \lambda I) = \left(-k - \frac{h}{q}\,U - \lambda\right)(h\,V - w - \lambda) + \frac{h^2}{q}\,V\,U. \tag{4.7}$$

Therefore, the corresponding characteristic equation is

$$\lambda^2 - \left(h\,V - w - k - \frac{h}{q}\,U\right)\lambda + kw - kh\,V + \frac{hw}{q}\,U = 0. \tag{4.8}$$

4.2.1 Analyzing the First Equilibrium $E_1 = (V_1, U_1) = (0, 0)$

From (4.5), we have the Jacobian

$$J_i = \begin{bmatrix} -k - \frac{h}{q}\,U_i & -\frac{h}{q}\,V_i \\ h\,U_i & h\,V_i - w \end{bmatrix}, \tag{4.9}$$

where $i = 1,2,3$. When $E_1 = (V_1, U_1) = (0, 0)$, the Jacobian becomes

$$J_1 = \begin{bmatrix} -k & 0 \\ 0 & -w \end{bmatrix}, \tag{4.10}$$

Hence, the corresponding characteristic equation of the steady state E_1 is

$$\lambda^2 + (w + k)\,\lambda + kw = 0. \tag{4.11}$$

Therefore, we have the following results.

Theorem 1. Let $k, h, q, w > 0$. The equilibrium $E_1 = (V_1, U_1) = (0, 0)$ of the model (4.1) is a stable node.

Proof. By (4.10), we have

$$J_1 = \begin{bmatrix} -k & 0 \\ 0 & -w \end{bmatrix}. \tag{4.13}$$

(4.13) shows that the trace is $Tr(J_1) = -(k + w)$, and the determinant is $det(J_1) = kw$. Since $k, h, q, w > 0$, the trace is always negative, and the determinant is always positive. Therefore, the equilibrium $E_1 = (V_1, U_1) = (0, 0)$ is a stable node.

4.2.2 Analyzing the Second Equilibrium $E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$.

We have the Jacobian

$$J_i = \begin{bmatrix} -k - \frac{h}{q}\,U_i & -\frac{h}{q}\,V_i \\ h\,U_i & h\,V_i - w \end{bmatrix}, \tag{4.14}$$

where $i = 1,2,3$. So, the Jacobian of $E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$ is

$$J_2 = \begin{bmatrix} -k & -\dfrac{h\varphi}{qk} \\ 0 & \dfrac{h\varphi}{k} - w \end{bmatrix}. \tag{4.15}$$

The characteristic equation of the steady state $E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$ is equivalent to

$$\lambda^2 + \left(k - \frac{h\varphi}{k} + w\right)\lambda - h\,\varphi + kw = 0.$$

Therefore, the trace of J_2 is $Tr(J_2) = -k + \frac{h\varphi}{k} - w$, and the determinant is $det(J_2) = kw - h\,\varphi$. This gives the following Theorem.

Theorem 2. Let $k, h, q, w > 0$. The equilibrium $E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$ of the model (4.1) is

1. a stable node if $h\,\varphi < k^2 + kw$, and $h\,\varphi < kw$.
2. s saddle point if $h\,\varphi > kw$.

4.2.3 Analyzing the Third Equilibrium $\boldsymbol{E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)}$

We find that the Jacobian matrix to be

$$J_i = \begin{bmatrix} -k - \dfrac{h}{q}\,U_i & -\dfrac{h}{q}\,V_i \\ h\,U_i & h\,V_i - w \end{bmatrix}, \tag{4.16}$$

where $i = 1,2,3$. When $\boldsymbol{E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)}$, the Jacobian matrix becomes

$$J_i = \begin{bmatrix} -k - \dfrac{h\,\varphi}{w} - k & -\dfrac{w}{q} \\ \dfrac{hq\varphi}{w} + kq & 0 \end{bmatrix}. \tag{4.17}$$

The characteristic equation of the steady state $\boldsymbol{E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)}$ **is**

$$\lambda^2 + \left(k + \frac{h\varphi}{w} + kh\right)\lambda + h\,\varphi + kw = 0. \tag{4.18}$$

Theorem 3. Let $k, h, q, w > 0$. The equilibrium $E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)$ of the model (4.1) is a stable node.

Proof. By (4.17), we have

$$J_i = \begin{bmatrix} -2k - \dfrac{h\,\varphi}{w} & -\dfrac{w}{q} \\ \dfrac{hq\varphi}{w} + kq & 0 \end{bmatrix}. \tag{4.19}$$

(4.13) shows that the trace is $Tr(J_3) = -2k - \frac{h\,\varphi}{w}$, and the determinant is $det(J_3) = \frac{w}{q}\left(\frac{hq\varphi}{w} + kq\right) = h\varphi + wk$ .nce $k, h, q, w > 0$, the trace is always negative, and the determinant is always positive. Therefore, the third equilibrium $E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)$ is a stable node.

The equilibria and their stability conditions are summarized in Table 1.

The Equilibria	Stability Conditions
$E_1 = (V_1, U_1) = (0, 0)$	Always stable.
$E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$	Stable if $h\,\varphi < k^2 + kw$, and $h\,\varphi < kw$.
	Saddle point if $h\,\varphi > kw$.
$E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)$	Always stable

Table 1. Equilibria and their stability conditions. Here, $k, h, q, w > 0$.

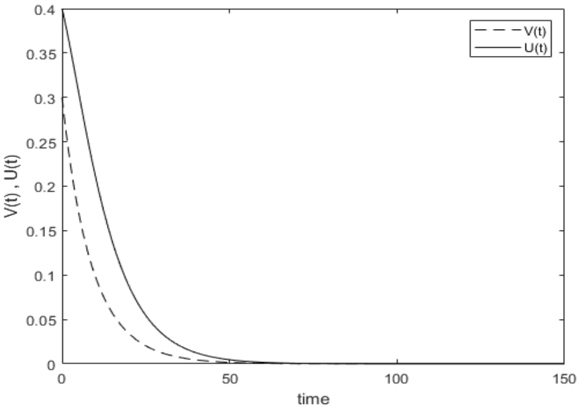
5. Numerical Analysis and Simulations

5.1 Numerical solutions of the proposed model

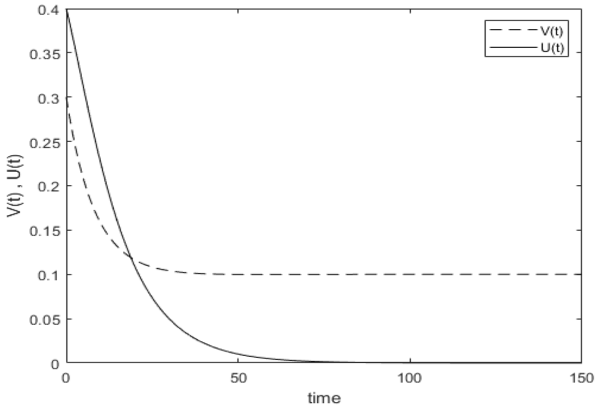
In this part, we seek the numerical solution of the system (2.1) by assuming h is a constant, i.e., we will solve the following model

$$\begin{aligned} \frac{dV(t)}{dt} &= \varphi - k\,V(t) - \frac{h}{q}\,V(t)U(t), \\ \frac{dU(t)}{dt} &= h\,V(t)\,U(t) - w\,U(t), \end{aligned} \tag{5.1}$$

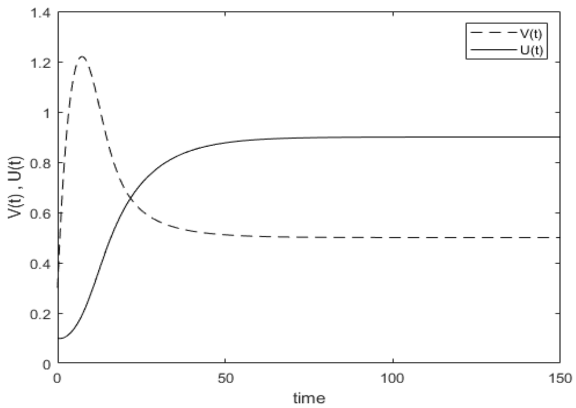
where $t \in [0, 150]$. The numerical values of the parameters k, h, q, w are chosen differently. The numerical solutions of the model (5.1) are demonstrated in Fig. 1, Fig. 2, and Fig. 3.



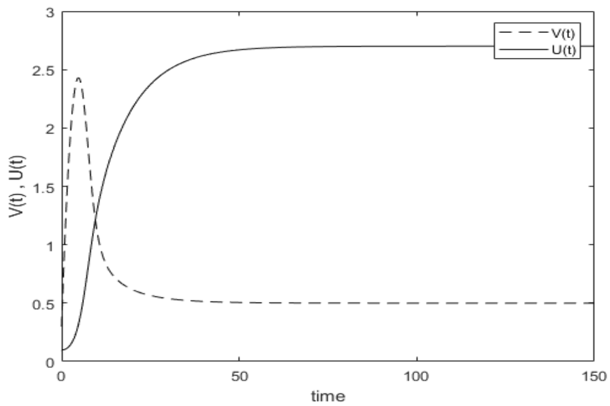
(a) $\varphi = 0$



(b) $\varphi = 0.01$

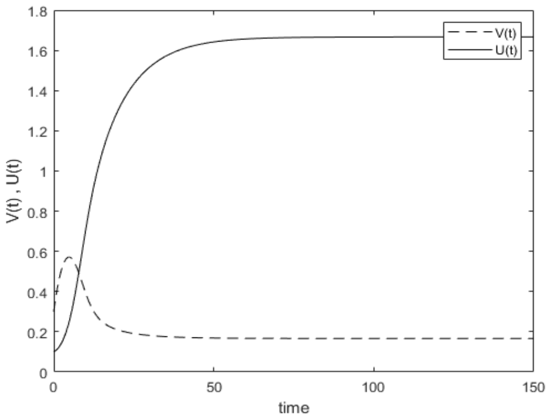


(c) $\varphi = 0.4$

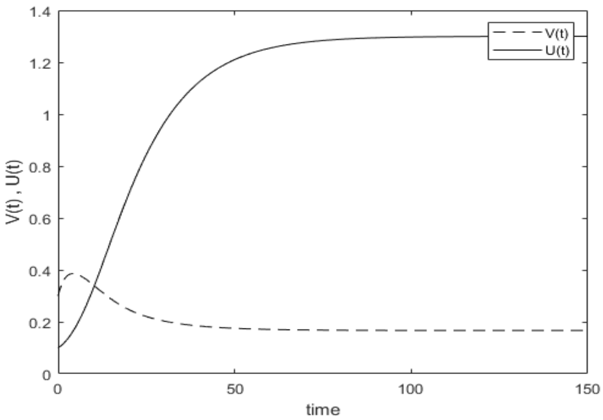


(d) $\varphi = 1$

Figure 1: The resource and consumer model dynamics at different values of φ. The dashed line is the biomass of food resources V, while the solid line is the consumer biomass of the population U.

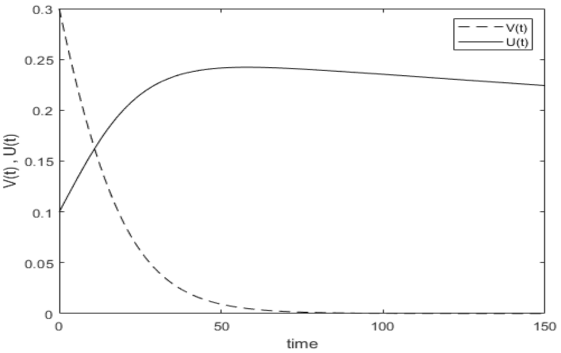


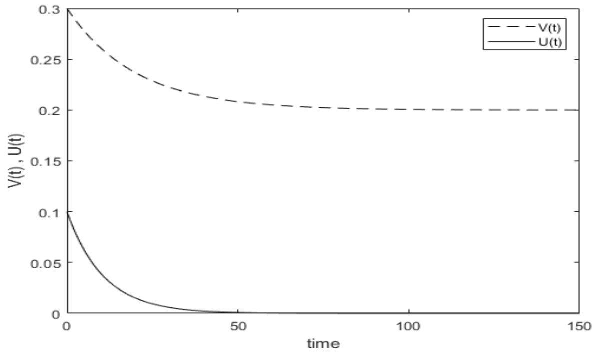
(a) $\boldsymbol{k = 0.2}$



(b) $\boldsymbol{k = 0.42}$

Figure 2: The resource and consumer model dynamics at different values of k. The dashed line is the biomass of food resources V, while the solid line is the consumer biomass of the population U. Other values are $\varphi = 0.2$, $h = 0.6, q = 1$, $w = 0.1$.





(a) **(b)**

Figure 3: The numerical solutions of the resource-consumer model. The dashed line is the biomass of food resources V, while the solid line is the consumer biomass of the population U. Here, (a) $\varphi = 0.00001, k = 0.03, h = 0.2, q = 1, w = 0.001$. (b) $\varphi = 0.01, k = 0.05, h = 0.02, q = 100, w = 0.1$.

5.2 Numerical analysis of the steady states

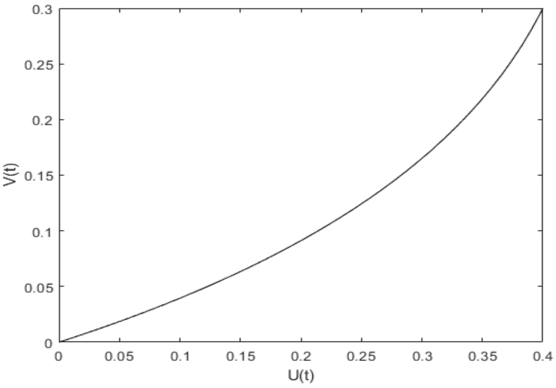
This section discusses the resulting dynamics of the steady states. The model we solve numerically is

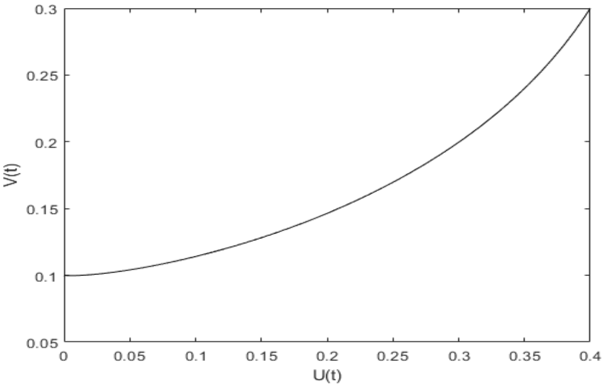
$$\frac{dV(t)}{dt} = \varphi - k\,V(t) - \frac{h}{q}\,V(t)U(t),$$
$$\frac{dU(t)}{dt} = h\,V(t)\,U(t) - w\,U(t). \tag{5.2}$$

The steady-state solutions of (5.2) are given in Fig. 4, Fig. 5, and Fig. 6. Fig. 4, Fig. 5, and Fig. 6. demonstrate Phase-plots of the biomass of food resources $V(t)$ against the consumers' biomass of the population $U(t)$. Assuming $\varphi = 0$ leads to decreased consumers' biomass due to the lack of food resources as in Fig. 4 (a). In this case, the biomass of food resources $V(t)$ and the consumers' biomass of the population $U(t)$go to zero with time, which shows that the equilibria $E^* = (0, 0)$ is locally asymptotically stable. In case $\varphi = 0.01$ the consumer biomass of the population $U(t)$ reduces to zero, while the biomass of food resources $V(t)$ converges to 0.1. With this option, the equilibria $E^* = (V^*, U^*) = (0.1, 0)$ is also locally asymptotically stable as in Fig. 4(b).

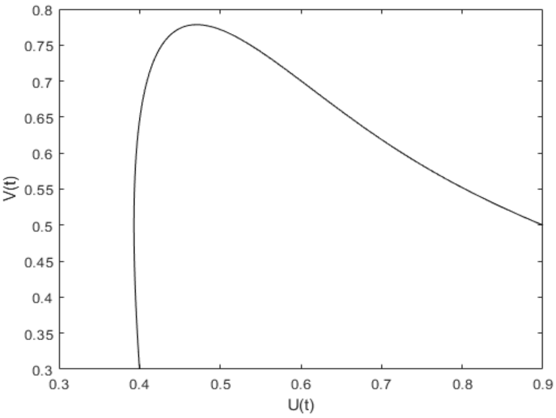
When $0.4 \le \varphi \le 1$, the interior equilibria are locally asymptotically stable as given in Fig. 4(c)(d). Fig. 5 displays that both the biomass of food resources $V(t)$ and the consumers' biomass of the population $U(t)$ will reach the positive interior equilibrium, which is locally asymptotically stable in this case.

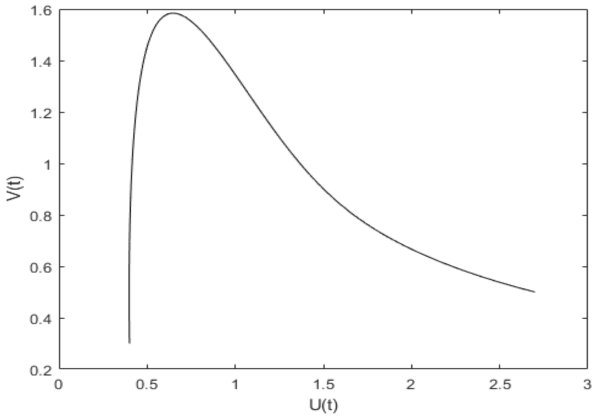
Fig. 6(a) illustrates that the biomass of food resources $V(t)$ converges to zero, while the consumer biomass of the population $U(t)$ approaches 0.2. Therefore, the equilibria $E^* = (V^*, U^*) = (0, 0.2)$ is locally asymptotically stable. Fig. 6(b) indicates that the biomass of food resources $V(t)$ converges to 0.2, while the consumers' biomass of the population $U(t)$ approaches zero. We conclude that the interior equilibria $E^* = (V^*, U^*) = (0.2, 0)$ is locally asymptotically stable.





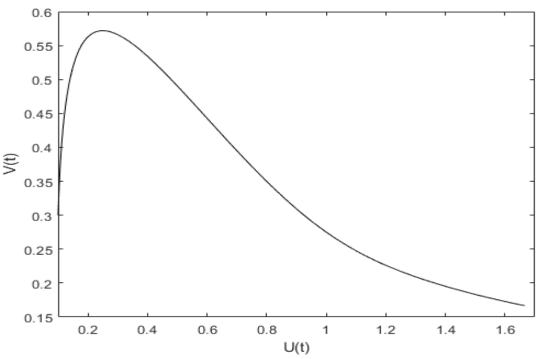
(a) $\varphi = 0$ (b) $\varphi = 0.01$

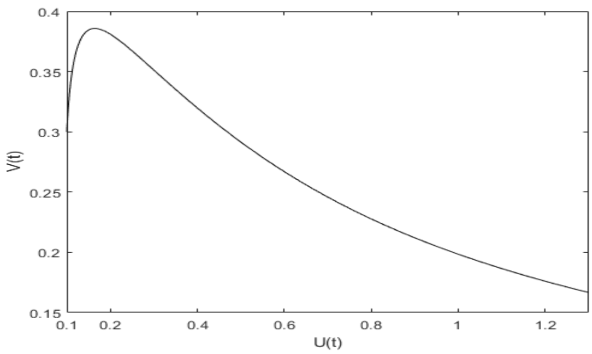




(c) $\varphi = 0.4$ (d) $\varphi = 1$

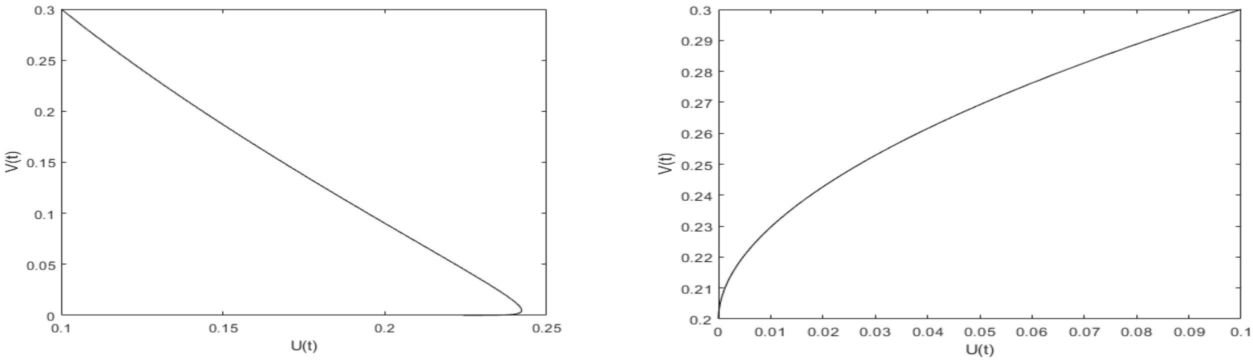
Figure 4: Phase-plots of the biomass of food resources $V(t)$ over the consumers biomass of the population $U(t)$ at different values of φ.





(a) $\boldsymbol{k = 0.2}$ **(b)** $\boldsymbol{k = 0.42}$

Figure 5: The resource and consumer model dynamics at different values of k. The dashed line is the biomass of food resources V, while the solid line is the consumer biomass of the population U. Other values are $\varphi = 0.2$, $h = 0.6, q = 1$, $w = 0.1$.



(a) **(b)**

Figure 6: The numerical solutions of the resource-consumer model. The dashed line is the biomass of food resources V, while the solid line is the consumer biomass of the population U. Here, (a) $\varphi = 0.00001, k = 0.03, h = 0.2, q = 1, w = 0.001$. (b) $\varphi = 0.01, k = 0.05, h = 0.02, q = 100, w = 0.1$.

6. Discussion and conclusion

The article examines a mathematical model that contains food (the resources) and the population (the consumer). The analytic and numerical results verify the existence of the solutions. When there are not enough resources, i.e., $\varphi \to 0$, the biomass of the resources V and the biomass of the consumers U converge to zero, as in Fig. 1(a)(b) and Fig. 4(a)(b). This situation leads to the first equilibrium $E_1 = (V_1, U_1) = (0, 0)$, where the biomass of the resources V and the biomass of the consumers U collapse. As we increase the amount of food provided to the population, we see that biomass of the resources and consumers approach a positive steady state, as demonstrated in Fig. 1(c)(d) and Fig. 2(a)(b). In this scenario, the biomass of the resources V and the biomass of the consumers U coexist.

Fig. 3 (a) illustrates that the consumer's biomass could temporarily grow when there are not enough resources. If $q \to \infty$, the consumer's biomass plunges to zero while the resource biomass continues to exist as $t \to \infty$, as reflected in Fig. 3(b).

This paper analytically shows the existence of steady states. The numerical dynamics of the steady-state solutions are provided in Section 5. By (Theorem 1), The equilibrium $E_1 = (V_1, U_1) = (0, 0)$ is a stable node if we have positive parameters of the proposed model. The second steady state $E_2 = (V_2, U_2) = \left(\frac{\varphi}{k}, 0\right)$ is either a stable node or saddle point as discussed in Theorem 2. Theorem 3 reveals that the equilibrium $E_3 = (V_3, U_3) = \left(\frac{w}{h}, \frac{q\varphi}{w} + \frac{kq}{h}\right)$ is a stable node. The equilibrium points and their stability conditions are summarized in Table 1. These results are proven based on the results and conditions provided by Leah [4].

In the future, we will study the model with delay to further investigate the dynamics. We will also discuss the influence of $h(t)$ as a function of time on the system's dynamics.

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