Journal of Qassim University for Science
2025, Vol 4 (Issue 1), pp 183-192

RECENT DEVELPOMENTS ON GROMOV-WITTEN THEORY OF HILBERT SCHEMES OF POINTS ON CERTAIN SURFACES

MAZEN M ALHWAIMEL

ABSTRACT. In this paper, we survey recent progress on the theory of Gromov-Witten invariants on Hilbert schemes of points mainly on elliptic surfaces and simply connected minimal surface of general type. In particular, we focus on the aspects of computational progress that has been done in the cases of lower genus. Then, we discuss some important conjectures that have been proposed and gather all available information and progress toward answering them.

1. Introduction

Hilbert schemes are fundamental structures in algebraic geometry. They allow the study of certain family of geometric objects in a given variety [6, 19]. Specifically, Hilbert schemes of points parametrize zero dimensional closed subschemes in a given scheme. This turns out to be a very significant structure that can be used to study problems in enumerative geometry. On the other hand, Gromov-Witten invariants are topological invariants that count the number of curves passing through a fixed set of points. Thus, studying curves passing through a given set of points can be viewed as a study of geometry of Hilbert schemes as parameter spaces of those fixed points. In general, Hilbert schemes of points can develop all sort of singularities on a given schemes, but for the case of surfaces they are always smooth [4, 5]. The first progress toward study of 1-point Gromov-Witten invariants for Hilbert schemes on surfaces was done in [15]. To briefly lay down their method: let $X^{[n]}$ be the Hilbert scheme of points on a smiply-connected surface X that is smooth and projective. We view elements of $X^{[n]}$ as length-n 0-dimensional closed subschemes ξ of X. Let $x_1, \cdots, x_{n-1} \in X$ be distinct fixed points. Put

$$M_2(x_1) = \{\xi \in X^{[2]} | \mathrm{Supp}(\xi) = \{x_1\}\}.$$

$M_2(x_1)$ is the punctual Hilbert scheme parametrizing length-2 0-dimensional closed subschemes supported at x_1 which is known to be isomorphic to the projective line $\mathbb{P}^1$. Smooth rational curves in $X^{[n]}$ are described as

$$\beta_n =: \{ \xi + x_2 + \cdots + x_{n-1} \in X^{[n]} | \ \xi \in M_2(x_1)\}. \tag{1.1}$$

It is clear that these curves are mapped to points by the map

$$\rho_n : X^{[n]} \to X^{(n)},$$

Date: April 26, 2025.
2020 *Mathematics Subject Classification.* Primary 14C05; Secondary 14N35.
Key words and phrases. Gromov-Witten invariants, Hilbert schemes, cosection localization.

which is the Hilbert-Chow morphism that sends elements in the Hilbert scheme $X^{[n]}$ to their support on the n-th symmetric product $X^{(n)}$ [17, 18].

For an integer $d > 0$, let $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, d\beta_n)$ be the moduli space of r-pointed stable maps $\mu : (D; q_1, \cdots, q_r) \to X^{[n]}$ where D is a nodal curve of genus g and $q_1, \cdots q_r \in D$ are distinct smooth points. It is known that the expected dimension of this moduli spaces is given by

$$\mathfrak{d} = -(K_{X^{[n]}} \cdot d\beta_n) + (\dim(X^{[n]}) - 3)(1 - g) + r = (2n - 3) \cdot (1 - g) + r.$$

Let $\gamma \in H^{4n-4}(X^{[n]}, \mathbb{C})$ be a cohomology class, and let

$$\mathrm{ev}_j : \overline{\mathfrak{M}}_{g,r}(X^{[n]}, d\beta_n) \to X^{[n]}$$

be the evaluation map defined by $\mathrm{ev}_j([\mu : (D; q_1, \cdots, q_r) \to X^{[n]}]) = \mu(q_j)$. For the virtual fundemental class $[\overline{\mathfrak{M}}_{g,1}(X^{[n]}, d\beta_n)]^{\mathrm{vir}}$, the 1-point Gromov-Witten invariant is defined to be

$$\langle \gamma \rangle_{g,d\beta_n}^{X^{[n]}} = \int_{[\overline{\mathfrak{M}}_{g,1}(X^{[n]}, d\beta_n)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma).$$

When X is a simply-connected smooth projective surface and motivated by Ruan's conjecture for a crepant resolution for the Hilbert-Chow morphism defined above, Li and Qin made an investigation on the 1-point genus-0 Gromov-Witten invariants of the crepant resolution ρ_n. They analyzed the obstruction bundle over the moduli space $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ and reduced their computation to only curves of the type described in (1.1). As a result, they were able to compute the 1-point genus-0 Gromov-Witten invariants $\langle \gamma \rangle_{0,d\beta_n}^{X^{[n]}}$ in Theorem 3.5 [15].

In this article, we will mainly focus on the case when X is an elliptic surface, in particular, the case $X = C \times E$ for some smooth projective curve C and some elliptic curve E.

The article is orgenized as followes. In Section 2, we go over some main ideas and definitions which are needed in the rest of this article. In Sec 3, we go over main vanishing theorems detailing the reason for considering only specific cases that give non-trivial contribution. In Section 4, we focus on a special type of elliptic surfaces and put together most up to date calculation of the 1-point Gromo-Witten invariants. We split this section into two subsections detailing techniques used to tackle two cases depending on the genus. In Section 5, we collect the important results for the case of minimal surfaces of general type.

Conventions: In this paper, an elliptic surface means a smooth projective surface which admits an elliptic fiberation over a smooth curve and is relative minimal. For a smooth projective surface X, let $q = h^1(X, \mathcal{O}_X)$ and

$$p_g = h^2(X, \mathcal{O}_X) = h^0\big(X, \mathcal{O}_X(K_X)\big).$$

2. **Quick review of stable maps and Gromov-Witten invariants**

Let Y be a smooth projective variety. An r-pointed stable map to Y is defined as a complete nodal curve D with r distinct ordered smooth points $p_1, \ldots, p_r$ along with a morphism $\mu : D \to Y$ such that the data $(\mu, D, p_1, \ldots, p_r)$ has only finitely many automorphisms. We denote this stable map by

$$[\mu : (D; p_1, \ldots, p_r) \to Y],$$

or simply $[\mu : D \to Y]$. For $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ be the coarse moduli space parametrizing the stable maps $[\mu : (D; p_1, \ldots, p_r) \to Y]$ such that $\mu_*[D] = \beta$ and the arithmetic genus of D is g. We define the i-th evaluation map as follows:

$$\mathrm{ev}_i \colon \overline{\mathfrak{M}}_{g,r}(Y, \beta) \to Y, \tag{2.1}$$

which is given by $\mathrm{ev}_i([\mu : (D; p_1, \ldots, p_r) \to Y]) = \mu(p_i)$. It is known [12, 13, 3] that the coarse moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ is projective and has a virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}} \in A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$ where

$$\mathfrak{d} = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r, \tag{2.2}$$

is the expected complex dimension of $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$, and $A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$ denotes the Chow group of $\mathfrak{d}$-dimensional cycles in the moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$.

The Gromov-Witten invariants are defined by using the virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}}$. An element

$$\alpha \in H^*(Y, \mathbb{C}) \stackrel{\mathrm{def}}{=} \bigoplus_{j=0}^{2\dim_{\mathbb{C}}(Y)} H^j(Y, \mathbb{C}),$$

is referred to as *homogeneous* if $\alpha \in H^j(Y, \mathbb{C})$ for some j; in which case, we assign $|\alpha| = j$. Let $\alpha_1, \ldots, \alpha_r \in H^*(Y, \mathbb{C})$ be homogeneous such that

$$\sum_{i=1}^{r} |\alpha_i| = 2\mathfrak{d}. \tag{2.3}$$

The r-point Gromov-Witten invariant is then defined by

$$\langle \alpha_1, \ldots, \alpha_r \rangle_{g,\beta}^{Y} = \int_{[\overline{\mathfrak{M}}_{g,r}(Y,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1) \otimes \ldots \otimes \mathrm{ev}_r^*(\alpha_r). \tag{2.4}$$

In particular, when $r = 1$, we see from the projection formula that

$$\langle \alpha \rangle_{g,\beta}^{Y} = \int_{\mathrm{ev}_{1*}([\overline{\mathfrak{M}}_{g,1}(Y,\beta)]^{\mathrm{vir}})} \alpha. \tag{2.5}$$

For $0 \le i < r$, we shall use

$$f_{r,i} : \overline{\mathfrak{M}}_{g,r}(Y, \beta) \to \overline{\mathfrak{M}}_{g,i}(Y, \beta) \tag{2.6}$$

to stand for the forgetful map obtained by forgetting the last $(r - i)$ marked points and contracting all the unstable components.

3. **Vanishing theorems**

The Vanishing Theorems developed in [8, 9, 14, 11, 10, 20] for Gromov-Witten invariants provide a powerful tool for reducing cases in the study of Gromov-Witten theory. The theorems assert that under certain conditions—such as when the genus of the curve or the degree of the curve is sufficiently large—the Gromov-Witten invariants of a variety vanish. This allows one to eliminate many potentially non-trivial cases, focusing only on the relevant, non-vanishing contributions. By systematically applying these theorems and their corollaries, one can simplify the analysis of Gromov-Witten invariants and reduce the complexity of problems, making it easier to extract meaningful results from the moduli spaces of stable maps and their associated counts of curves. We will briefly go over some of the main theorems that have been developed in the aforementioned papers and book.

Let X be a smooth projective complex surface. For simplicity, put

$$\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta).$$

Assume that the surface X admits a non-trivial holomorphic 2-form

$$\theta \in H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X)).$$

It is known that θ induces a holomorphic 2-form $\theta^{[n]}$ of the Hilbert scheme $X^{[n]}$ which can also be regarded as a map $\theta^{[n]} : T_{X^{[n]}} \to \Omega_{X^{[n]}}$. In turn, $\theta^{[n]}$ induces a regular cosection

$$\sigma : \quad \mathcal{O}b_{\overline{\mathfrak{M}}} \longrightarrow \mathcal{O}_{\overline{\mathfrak{M}}} \tag{3.1}$$

of the obstruction sheaf $\mathcal{O}b_{\overline{\mathfrak{M}}}$ of $\overline{\mathfrak{M}}$. The degeneracy locus

$$\overline{\mathfrak{M}}(\sigma) \tag{3.2}$$

of σ is the subset of $\overline{\mathfrak{M}}$ consisting of all the stable maps $u : \Gamma \to X^{[n]}$ such that the composition

$$u^*(\theta^{[n]}) \circ du : \quad T_{\Gamma_{\mathrm{reg}}} \to u^*T_{X^{[n]}}|_{\Gamma_{\mathrm{reg}}} \to u^*\Omega_{X^{[n]}}|_{\Gamma_{\mathrm{reg}}} \tag{3.3}$$

is trivial over the regular locus Γ_{reg} of Γ.

Theorem 3.1. *(*[20, Theorem 13.6]*) Let X be a simply connected smooth projective complex surface admitting a holomorphic 2-form θ, and let $C_{0,1}, \ldots, C_{0,s}$ be the irreducible components (with reduced scheme structures) of the zero divisor of θ. If*

$$\beta \neq \sum_{i=1}^{s} d_i \beta_{C_{0,i}} - d\beta_n$$

for some integers $d_1, \ldots, d_s \geq 0$ and d, then all the Gromov-Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish.

Proof. It is known that the degeneracy locus $\overline{\mathfrak{M}}(\sigma)$ from (3.2) is empty ([20, Lemma 13.5]). It follows from that

$$[\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)]^{\mathrm{vir}} = 0.$$

Therefore, all the Gromov-Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish. □

Corollary 3.2. *([20, Corollary 13.7]) Let X be a simply connected smooth projective complex surface admitting a holomorphic 2-form with irreducible zero divisor. If $\beta \neq d_0\beta_{K_X} - d\beta_n$ for some integer d and rational number $d_0 \geq 0$, then all the Gromov-Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish.*

Proof. Let $\theta \in H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X))$ be the holomorphic 2-form whose zero divisor C_0 is irreducible (but possibly non-reduced). Then,

$$K_X = C_0 = m(C_0)_{\mathrm{red}} \tag{3.4}$$

for some positive integer m, and the corollary follows from Theorem 3.1. □

Corollary 3.3. *([20, Corollary 13.9]) Let X be a simply connected (minimal) elliptic surface without multiple fibers and with positive geometric genus. Let $n \geq 2$ and $\beta \neq 0$. Then all the Gromov-Witten invariants without descendant insertions defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish, except possibly when $0 \leq g \leq 1$ and $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer d and rational number $d_0 \geq 0$.*

Corollary 3.4. *([20, Corollary 13.10]) Let X be a simply connected minimal surface of general type admitting a holomorphic 2-form with irreducible zero divisor. Let $n \geq 2$ and $\beta \neq 0$. Then all the Gromov-Witten invariants without descendant insertions defined via $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish, except possibly in the following cases*

(i) *$g = 0$ and $\beta = d\beta_n$ for some integer $d > 0$;*
(ii) *$g = 1$ and $\beta = d\beta_n$ for some integer $d > 0$;*
(iii) *$g = 0$ and $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer d and rational number $d_0 > 0$.*

Proof. In view of Corollary 3.2, it remains to consider the case when $\beta = d_0\beta_{K_X} - d\beta_n$ for some integer d and rational number $d_0 \geq 0$.

When $d_0 = 0$ and $\beta = d\beta_n$ with $d > 0$, and using the fact that the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ is

$$\mathfrak{d} = -K_{X^{[n]}} \cdot \beta + (\dim X^{[n]} - 3)(1 - g) + r, \tag{3.5}$$

we see that the expected dimension of the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ is equal to

$$\mathfrak{d} = (2n - 3)(1 - g) + r.$$

If $g \geq 2$, then all the Gromov-Witten invariants without descendant insertions defined via $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish by the Fundamental Class Axiom.

Next, assume that $d_0 > 0$. Since $K_X^2 \geq 1$, we see from (3.5) that

$$\mathfrak{d} < (2n - 3)(1 - g) + r.$$

By the Fundamental Class Axiom, all the Gromov-Witten invariants without descendant insertions vanish except possibly in the case when $g = 0$. □

4. The case for $X^{[2]}$ when X is an eliptic surface

Based on the analysis given in the previous section, we see that we can restrict our attention to only the cases where the Gromov-Witten invariants do not vanish. In this section, we collect all these important cases and provide a general look on how we prove them. We also show that these special cases are positive answers to more general open conjectures.

4.1. **When** $g = 0$**.** We start this part by the following lemma, which we need to establish some notations and also will be used to prove some theorems.

Lemma 4.1. *([2, Lemma 5.5]) Let $X = C \times E$ where E is an elliptic curve and C is a smooth curve. Let $p_2 : C \times X^{[2]} \to X^{[2]}$, $\tilde{p}_1 : C \times E^{(2)} \to C$ and $\tilde{p}_2 : C \times E^{(2)} \to E^{(2)}$ be the natural projections. Then we have the exact sequence*

$$0 \to \tilde{p}_2^* T_{E^{(2)}} \to p_2^* T_{X^{[2]}}|_{C \times E^{(2)}} \to \tilde{p}_1^* T_C \oplus \big(\tilde{p}_1^* T_C \otimes \tilde{p}_2^* \widetilde{L}^{-1}\big) \to 0 \tag{4.1}$$

where $\widetilde{L} = \mathcal{O}_{E^{(2)}}(2\Xi) \otimes (\mathrm{AJ})^* \mathcal{O}_{\mathbb{J}}(-\widetilde{\Theta})$ for some Theta divisor $\widetilde{\Theta}$ on $\mathbb{J} = \mathrm{Jac}_2(E)$, and $T_{E^{(2)}}$, $T_{X^{[2]}}$ and T_C are the tangent bundles of $E^{(2)}$, $X^{[2]}$ and C respectively.

By [2, (59)], there exists a bijective morphism

$$\Psi_{g,r} : \overline{\mathfrak{M}}_{g,r}(d) \to C \times \overline{\mathfrak{M}}_{g,r}\big(E^{(2)}, d[\tilde{f}]\big) \tag{4.2}$$

where $\overline{\mathfrak{M}}_{g,r}(d) = \overline{\mathfrak{M}}_{g,r}\big(X^{[2]}, d(\beta_f - 2\beta_2)\big)$ and $\tilde{f}$ denotes a fiber of the Abel-Jacobi map $\mathrm{AJ} : E^{(2)} \to \mathrm{Jac}_2(E)$.

Now, we state the first conjecture of this paper.

Conjecture 4.2. ([2, Conjecture 1.2]) Let $d > 0$. Let X be a smooth surface with an elliptic fibration $\pi : X \to C$ for some smooth curve C. If $K_X = \pi^* \kappa$ for some divisor κ, then

$$\mathrm{ev}_{1*}\Big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_f - 2\beta_2)\big)\big]^{\mathrm{vir}}\Big) = -\frac{\deg(\kappa)}{d^2} \cdot \big[f^{(2)}\big] \quad \in A_2(X^{[2]})$$

where f denotes (the class of) a smooth fiber of π.

This conjecture is still open in general. The only known case is Theorem 4.3 ([2, Theorem 5.7]). This special case is proved by using techniques of cosection localization of Kim and Li [9], analyzing the degeneracy locus and studying obstruction sheaf of the moduli space $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_f - 2\beta_2)$.

Theorem 4.3. *([2, Theorem 5.7]) Let $X = C \times E$ where E is an elliptic curve and C is a smooth curve. Let $d \geq 1$ and f be a fiber of the natural projection $X \to C$. Then,*

$$\mathrm{ev}_{1*}\Big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_f - 2\beta_2)\big)\big]^{\mathrm{vir}}\Big) = -\frac{2g_C - 2}{d^2} \cdot \big[f^{(2)}\big] \quad \in A_2(X^{[2]}).$$

In the following, we outline the proof of Theorem 4.3, which is broken down into smaller steps.

Step 1: Put $f = \{c\} \times E$ where $c \in C$. For simplicity, put

$$\begin{aligned} \mathcal{E} &= (\Psi_{0,0})^* (\mathrm{Id}_C \times \Phi)^* \mathbb{V} \otimes \tau_0^* \big(R^1 (\tilde{f}_{1,0})_* \widetilde{\mathrm{ev}}_1^* \mathcal{O}_{\mathbb{P}}(-1)\big) \\ &= (\Psi_{0,0})^* \big((\mathrm{Id}_C \times \Phi)^* \mathbb{V} \otimes \rho_2^* \big(R^1 (\tilde{f}_{1,0})_* \widetilde{\mathrm{ev}}_1^* \mathcal{O}_{\mathbb{P}}(-1)\big)\big) \end{aligned} \tag{4.3}$$

In [2], the authors established that

$$[\overline{\mathfrak{M}}_{0,1}(d)]^{\mathrm{vir}} = (2 - 2g_C) \cdot \mathrm{ev}_1^* \big[f^{(2)}\big] \cdot f_{1,0}^* \big(c_{2d-2}(\mathcal{E})\big). \tag{4.4}$$

By projection formula, we have

$$\mathrm{ev}_{1*}\big([\overline{\mathfrak{M}}_{0,1}(d)]^{\mathrm{vir}}\big) \quad = \quad (2-2g_C)\cdot\big[f^{(2)}\big]\cdot \mathrm{ev}_{1*}f_{1,0}^{*}\big(c_{2d-2}(\mathcal{E})\big). \tag{4.5}$$

Step 2: Let $m(d,X) = (2-2g_C)\cdot \mathrm{ev}_{1*}f_{1,0}^{*}\big(c_{2d-2}(\mathcal{E})\big)$. To determine this rational number $m(d,X)$, choose very ample curves C_1, C_2 in X such that f, C_1, C_2 are in general position. Let $f\cap C_1 = \{x_1,\dots,x_s\}$ and $f\cap C_2 = \{y_1,\dots,y_t\}$. Then, $s = \langle f, C_1\rangle$, $t = \langle f, C_2\rangle$, and $\mathfrak{a}_{-1}(C_1)\mathfrak{a}_{-1}(C_2)|0\rangle$ (where $\mathfrak{a}_{-1}$ denotes Heisenberg operator, for more detalied construction see [20, Chapter 3]) and $f^{(2)}\subset X^{[2]}$ intersect transversally at the points $\xi_{i,j} = x_i + y_j \in X^{[2]}$, $1\le i\le s$ and $1\le j\le t$. By (4.4), $[\overline{\mathfrak{M}}_{0,1}(d)]^{\mathrm{vir}}\cdot \mathrm{ev}_1^{*}\big(\mathfrak{a}_{-1}(C_1)\mathfrak{a}_{-1}(C_2)|0\rangle\big)$ is equal to

$$st(2-2g_C)\cdot \mathrm{ev}_1^{*}\big[\xi_{i,j}\big]\cdot f_{1,0}^{*}\big(c_{2d-2}(\mathcal{E})\big).$$

Step 3: Through various computations in ([2, Sec 5]), we have the following relation

$$\begin{aligned} &[\overline{\mathfrak{M}}_{0,1}(d)]^{\mathrm{vir}}\cdot \mathrm{ev}_1^{*}\big(\mathfrak{a}_{-1}(C_1)\mathfrak{a}_{-1}(C_2)|0\rangle\big)\\ = \;& \frac{1}{d^2}\cdot\langle f, C_1\rangle\cdot\langle f, C_2\rangle\cdot(2-2g_C). \end{aligned}\tag{4.6}$$

By ([2, Theorem 5.3]) and comparing with (4.5), we get $m(d,X) = (2-2g_C)/d^2$.

4.2. **When** $g=1$**.** For this case, we first state the following conjecture regarding genus-1 Gromov-Witten invariants for Hilbert schemes of two points on elliptic surfaces.

Conjecture 4.4. Let X be an elliptic surface without multiple fibers. Let $d\ge 1$ and f be a smooth fiber in X. Then,

$$\langle\rangle^{X^{[2]}}_{1,\, d(\beta_f-2\beta_2)} = (-1)^d\cdot\chi(X),$$

where $\chi(X)$ denotes the Euler characteristic of X.

In [1], the author has computed a special case for certain types of elliptic surfaces.

Theorem 4.5. *(*[1, Theorem 3.7]*) Let $X = C\times E$ where E is an elliptic curve and C is a smooth curve. Let $d\ge 1$ and f be a fiber of the natural projection $X\to C$. Then,*

$$\langle\rangle^{X^{[2]}}_{1,\, d(\beta_f-2\beta_2)} = 0. \tag{4.7}$$

The outline of the proof of Theorem 4.5 consists of expressing genus-1 Gromov-Witten invariant $\langle\rangle^{X^{[2]}}_{1,\, d(\beta_f-2\beta_2)}$ in terms of the obstruction sheaf

$$\mathcal{O}b = R^1(f_{1,0})_*\mathrm{ev}_1^{*}T_{X^{[2]}}. \tag{4.8}$$

Step 1: By ([1, Lemma 3.5]), it is established that

$$\langle\rangle^{X^{[2]}}_{1,\, d(\beta_f-2\beta_2)} = c_{2d+2}(\mathcal{O}b). \tag{4.9}$$

Step 2: Let $\mathcal{E} = R^1(f_{1,0})_*(\Psi_{1,1})^*(\mathrm{Id}_C \times \widetilde{\mathrm{ev}}_1)^*\big(\tilde{p}_1^*T_C \oplus \big(\tilde{p}_1^*T_C \otimes \tilde{p}_2^*\widetilde{L}^{-1}\big)\big)$. By the disscusion in the proof of ([1, Theorem 3.7]), we have

$$\langle\rangle^{X^{[2]}}_{1,\, d(\beta_f - 2\beta_2)} = -\lambda \cdot c_{2d+1}(\mathcal{E}),$$

where $\lambda = c_1(\mathcal{H})$, i.e. the first Chern class of the rank-1 Hodge bundle over $\overline{\mathfrak{M}}_{1,0}(d)$.

Step 3: Next, λ is computed by proving that the intersection of first Chern class of $\widetilde{\mathcal{H}}$, which is the Hodge bundle over $\overline{\mathfrak{M}}_{1,0}\big(E^{(2)}, d[\tilde{f}]\big)$, with $c_{2d}\big(\widetilde{\mathcal{E}} \otimes \Phi^*\mathcal{O}_{\mathbb{J}}(\widetilde{\Theta})\big)$ equals zero which establishes the theorem.

5. The case for $X^{[2]}$ when X is simply connected minimal surface of general type

Another known computations for simply connected minimal surface of general type have been done in the work of [14].

Theorem 5.1. *([14, Theorem 6.7]) Let X be a simply connected minimal surface of general type with $K_X^2 = 1$ and $1 \leq p_g \leq 2$ such that every member in $|K_X|$ is smooth. Then,*

$$\langle 1\rangle^{X^{[2]}}_{0,\, \beta_{K_X} - 3\beta_2} = (-1)^{\chi(\mathcal{O}_X)}.$$

To outline the proof, we need to consider two separate cases of p_g. Namely, $p_g = 1$ and $p_g = 2$.

Step 1: If X is simply connected surface, then the Euler characteristic $\chi(\mathcal{O}_X) = 1 + p_g$. Specifically, if arithmetic genus $p_g = 1$, $\overline{\mathfrak{M}}_{0,0}(X^{[2]}, \beta_{K_X} - 3\beta_2)$ is a smooth point; so $\langle 1\rangle^{X^{[2]}}_{0,\, \beta_{K_X} - 3\beta_2} = 1 = (-1)^2$ and the formula holds.

Step 2: Explicitly describe the line bundle $R^1 f_* \Psi^* T_{X^{[2]}}$, where $\mathrm{ev}_1 = \Psi$ is the evaluation map defined earlier. The description of this line bundle is

$$R^1 f_* \Psi^* T_{X^{[2]}} \cong \mathcal{O}_{|K_X|}(2) \otimes R^1 f_*\big(\tilde{p}_* \mathcal{O}_{\widetilde{X}}(E)\big), \tag{5.1}$$

where $E \subset \widetilde{X}$ is the exceptional curve of the blow up $\widetilde{X}$ of X.

Step 3: By (5.1), the authors explicitly descried the term $R^1 f_*\big(\tilde{p}_*\mathcal{O}_{\widetilde{X}}(E)\big)$. In order to do this, they determine the ruled surface $\mathcal{E}$ which is the exceptional divisor of the blowing-up of $\mathrm{Jac}_2(\mathcal{C}/|K_X|)$ along some section of the natural projection of $\mathrm{Jac}_2(\mathcal{C}/|K_X|)$ onto $|K_X|$ where $\mathcal{C} \subset |K_X| \times X$ is the family of curves parametrized by $|K_X|$.

Step 4: They proved that the ruled surface $\mathcal{E}$ is the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}\big(\mathcal{O}_{|K_X|} \oplus \mathcal{O}_{|K_X|}(-2)\big)$. By the Grothendieck-Riemann-Roch Theorem, and using some exact equences they established alomng the paper, they proved that

$$R^1 f_*\big(\tilde{p}_*\mathcal{O}_{\widetilde{X}}(E)\big) \cong \mathcal{O}_{|K_X|}(-5) \otimes \Big((f_*\mathcal{O}_{\mathcal{E}})^\vee \otimes \mathcal{O}_{|K_X|}(-2)^\vee\Big) \cong \mathcal{O}_{|K_X|}(-3). \tag{5.2}$$

Step 5: Combining (5.1) and (5.2), they obtain that

$$R^1 f_* \Psi^* T_{X^{[2]}} \cong \mathcal{O}_{|K_X|}(2) \otimes \mathcal{O}_{|K_X|}(-3) = \mathcal{O}_{|K_X|}(-1).$$

Step 6: By combining the last result with the following proposition, which we state here for completeness,

Proposition 5.2. *(*[14, Proposition 2.1]*)*
Let $\beta \in H_2(Y, \mathbb{Z})$ and $\beta \neq 0$. Let e be the excess dimension of the moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$. If $R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^*T_Y$ is a rank-e locally free sheaf over $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$, then $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ is smooth (as a stack) of dimension

$$\mathfrak{d} + e = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r + e, \tag{5.3}$$

and $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}} = c_e\big(R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^*T_Y\big) \cap [\overline{\mathfrak{M}}_{g,r}(Y, \beta)/\overline{\mathfrak{M}}_{g,r}]$.

they obtained that

$$\langle 1 \rangle^{X^{[2]}}_{0,\, \beta_{K_X} - 3\beta_2} = \deg\, [\overline{\mathfrak{M}}]^{\mathrm{vir}} = \deg\, c_1\big(R^1 f_* \Psi^* T_{X^{[2]}}\big) = -1. \qquad \square$$

.

6. **conclusion**

As discussed in the previous sections, we see promising opportunity to over come some of the difficulties arise during the computations. The Main difficulties when it comes to calculating Gromov-Witten invariants are the various techniques needed to carry out the computations. Another possible direction is to generalize to a another class of surfaces, and it would be interesting if any of the aforementioned theorems can be generalized to other types of surfaces.

References

[1] Mazem M. Alhwaimel, *Genus-1 Gromov-Witten invariants of Hilbert schemes of two points on elliptic surfaces.* preprint.

[2] M. M. Alhwaimel, Z. Qin, *Gromov-Witten invariants of Hilbert schemes of two points on elliptic surfaces.* Int. J. Math. **32** (2022), no. 10 & 11, 2250079, 27 pp.

[3] K. Behrend, B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997) 45-88.

[4] J. Fogarty, *Algebraic families on an algebraic surface.* Amer. J. Math. **90** (1968), 511-520.

[5] J. Fogarty, *Algebraic families on an algebraic surface. II: The Picard scheme of the punctual Hilbert scheme.* Amer. J. Math. **95** (1973), 660-687.

[6] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert.* Sem. Bourbaki **221**, 13. 1960-1961.

[7] J. Hu, W.-P. Li, Z. Qin, *The Gromov-Witten invariants of the Hilbert schemes of points on surfaces with $p_g > 0$.* Internat. J. Math. **26** (2015).

[8] Y. Kiem, J. Li, *Gromov-Witten invariants of varieties with holomorphic 2-forms.* Preprint.

[9] Y. Kiem, J. Li, *Localizing virtual cycles by cosections.* J. Amer. Math. Soc. **26** (2013), 1025-1050.

[10] J. Lee, T. Parker, *A structure theorem for the Gromov-Witten invariants of Kähler surfaces.* J. Differential Geom. **77** (2007), 483-513.

[11] J. Li, W.-P. Li, *Two point extremal Gromov-Witten invariants of Hilbert schemes of points on surfaces.* Math. Ann. **349** (2011), 839-869.

[12] J. Li, G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. A.M.S. **11** (1998), 19-174.

[13] J. Li, G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, (1998) 47–83.

[14] J. Hu, W.-P. Li, Z. Qin. *The Gromov-Witten invariants of the Hilbert schemes of points on surfaces with* $p_g > 0$. Internat. J. Math. 26 (2015), no. 1, 1550009, 26 pp.

[15] W.-P. Li, Z. Qin, *On* 1*-point Gromov-Witten invariants of the Hilbert schemes of points on surfacs*, Proceedings of 8th Gökova Geometry-Topology Conference (2001). Turkish J. Math. **26** (2002), 53-68.

[16] W.-P. Li, Z. Qin, *The Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphisms*. J. Differ. Geom. **104** (2016), 499-557.

[17] W.-P. Li, Z. Qin, Q. Zhang, *On the geometry of the Hilbert schemes of points in the projective plane*, preprint, math.AG/0105213.

[18] W.-P. Li, Z. Qin, Q. Zhang, *Curves in the Hilbert schemes of points on surfaces*. Contemp. Math. **322** (2003), 89-96.

[19] D. Mumford, *Towards an enumerative geometry of the moduli space of curves,* in Arithmetic and Geometry (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 1983, 271-328.

[20] Z. Qin, *Hilbert schemes of points and infinite dimensional Lie algebras*. Mathematical Surveys and Monographs **228**, 336pp, American Mathematical Society, Providence, RI, 2018.

Department of Mathematics, College of Science, Qassim University, P. O. Box 6644, Buraydah 51452, Saudi Arabia

Email address: 4200@qu.edu.sa